A mean field game approach to oil production

MFG R&D

Abstract

The following text integrates into one coherent exposition several contributions that have been published by Pierre-Louis Lions, Jean-Michel Lasry and Olivier Guéant in various places on the application of mean field games to model the production of a non-renewable good such as oil. These contributions are:

- "A mean field game application to economics: production of an exhaustible resource", published in "Paris-Princeton lectures in mathematical finance", Princeton Press, 2009
- "Mean field games and oil production" published in "Finance and Sustainable Development", Economica, 2009
- "Pic de Hubbert, rente de Hotelling et MFG", published in the proceedings of the conference "Printemps de la Chaire Finance et Développement Durable", May 6th 2009
- "Oil production and strategic interactions with substitutes: a Mean-Field Game approach", published in the proceedings of the Workshop "Optimization, Transport and Equilibrium in Economics" at the Ecole des Mines, July 6th 2009
- "Dynamique de production d'une ressource épuisable : une approche MFG" published in the proceedings of the conference "Défis actuels de la Finance", held at Université Paris 13, November 12th 2009.

Contents

1 Mean field games

1.1 Introduction

Mean field games (MFG) have been introduced by Jean-Michel Lasry and Pierre-Louis Lions [10, 11, 12, 13, 14] as a set of tools to model games with infinitely many players. These games are most often going to represent economic problems with a large number of agents and the mean field games tools will help solve the problem.

Mean field games theory does not introduce a new paradigm but rather what could be called a new "version" - or as they say in the software industry, a new "release" - of the existing paradigm in economics (utility maximization, rational expectations, ...). This new version is fully compatible with the preceding versions. As the software specialists put it: all the "functionality" of the previous "release" will still be there, and can be re-used in the context of the "update". Moreover, in this "MFG version", modeling can not only be re-applied as such, but it can be modified, extended and reworked to take advantage of the new "functionalities" provided by mean field games, i.e. by the capacity to introduce into the models the effects of social interaction between agents, which cannot be integrated into the framework of classical modeling in economics.
This compatibility with the classical paradigm is seen in the principles that mean field games re-adopt, in relation to defining (static or dynamic) equilibrium: rational expectations, invariance by permutation of similar agents, atomization of agents in a continuum to mention the classical principles found in a large part of economic theory (though not in all economic theory: in industrial economics, agents are not always permutable since names matter).
Another way of putting it is that mean field games constitute a mathematical toolbox that enables us to model effects which were sometimes ignored for lack of having the analytic and numerical tools needed to handle them. From the standpoint of such a toolbox, the new contribution of mean field games is given concrete expression in the mean field games system of HJB/Kolmogorov-type forward/backward equations that characterize equilibria. Let us now briefly

explain what this is.

In any (quantitative) modeling, there must always be an equation to express the optimization problem of each agent. Usually this involves one equation for each agent. If agents are grouped together by similar agent classes, there is one equation per agent class. This equation is generally a Bellman equation, since a large proportion of optimization problems falls within the framework of dynamic programming and of the various sophisticated forms of deterministic and stochastic control that are constructed and inscribed within the field of dynamic programming. Hence, Hamilton-Jacobi or the Hamilton-Jacobi-Bellman equations will be used to compute optimal behaviors.
An equation is also needed to express the group's behavior. When agents are atomized, the group is represented in the modeling by a distribution on the state space; or, if there are several agent classes, by a distribution for each class. The dynamics of the distribution is governed by a transport equation that can be called a Kolmogorov equation. In this transport equation, the optimal behaviors of agents occur as data, since it is the infinite collection (the continuum) of individual behaviors that is aggregated and which, thus aggregated, constitutes collective behavior.
Thus, the modeling of the behavior of a group of agents naturally leads to an HJB/K (Hamilton-Jacobi-Bellman and Kolmogorov) equations system. This HJB/K system is not new: it forms part of previous "versions". Admittedly, for technical reasons, it has not been much used. This is because in the case where the state space is continuous, a system of non-linear partial differential equations is not for the moment an attractive tool for most modelers (apart from specialist mathematicians). Most of the time, modelers prefer to circumvent this formalism by the various mathematical techniques that are available in each specific case. However, some authors make excellent use of it for certain types of problem.

But we have not yet answered the central questions: what is new in mean field games formalism? Essentially one thing: the density of agents on the state space can enter in the HJB equation.
Its novelty lies here. MFG equilibrium is defined by an HJB/K couple in

which the HJB and Kolmogorov equations are doubly coupled: individual behaviors are given for the Kolmogorov equation (this is not new) and, at the same time, the distribution of agents in the state space enters in the HJB equation (which is new).
This means that agents can incorporate into their preferences the density of states of other agents at the anticipated equilibrium. Therefore each agent can construct his strategy by taking account of the anticipated distribution of strategies and of the contingent decisions of other agents. Thus formulated, MFG equilibria can be seen as similar to another paradigm, that of Nash equilibria and more specifically of Isaacs-Bensoussan equilibria in n-player stochastic differential games. In fact, the MFG equilibria can be defined by moving to the limit on the number n of players in the class of differential games that are invariant by permutation of similar agents. It is this invariance by permutation (accompanied by a little continuity in the preference functions) which makes evident the distribution of agents in the space state as the object representing other agents for each atomized agent in the move to the limit.

Of course in games with a small number of players, permutation invariance is not a useful hypothesis: there are never two identical players around a poker table. Conversely, permutation invariance is very natural in most studies of large groups, especially in economics. On the other hand, the atomization of agents simplifies the strategic interactions: coughing on the part of one player can have major consequences in poker, but has no significance in a group of several thousand agents. This is why MFG equilibria equations are much easier to solve than Isaacs-Bensoussan equations. In other words, there is some hope of solving mean field games systems, and we can already do so for large classes (and we possess the corresponding numerical techniques), whereas the equations of n-player differential games are in general irresolvable since they describe situations in which the combinatory of strategic interactions exceeds calculation capacities.
These general considerations will be illustrated in what follows through application to an economic problem of producing an exhaustible resource.

1.2 Application to oil production

If one considers continuous time mean field games with a continuous state space, an example of application is that of the production of an exhaustible resource by a continuum of producers. If we suppose indeed for now that the number of producers is large, no producer can individually move the price in a given direction through his own production and hence the mean field hypothesis is suited to this pure competitive framework. Also, the forward/backward essence of mean field games is at work when it comes to consider the production of a resource whose availability is limited. The backward dimension comes indeed from the optimization of the production timing and the forward dimension is directly linked to the evolution of individual and global oil reserves.

In what follows, we are first going to consider as a reference case a more classical approach. With this approach, that is completely deterministic and can be tackled with Euler-Lagrange tools, we will derive results on oil production evolution, exhibit the Hotelling rent and discuss the presence of a Hubbert-like Peak. To generalize this first model we present a new mean field game approach we regard as more suited to the consideration of additional effects such as externality for instance. This mean field game approach is first considered a generalization of the deterministic model in a stochastic framework. Hence the first mean field game we present is just a basic model onto which other models can be grafted.
The introduction of the two partial differential equations that are at the core of the mean field game theory will allow for generalizations in various directions. For instance, the partial differential equations can be slightly modified to take account of producers' willingness to exploit the largest part of their reserves before other oil producers to avoid being one of the last oil producers: a situation that may be cumbersome because of the inherent randomness associated to the end of the oil era. Our model is also a basis for studying in greater depth the influence of the potential entry of new competitors, particularly those who are developing alternative energy sources (that are not exhaustible). This framework allows for instance to consider with powerful

analytical tools the negative effect in terms of carbon emissions of a subsidy to alternative energy producers as in [3].

2 The deterministic model

2.1 Basis of the model

2.1.1 Introduction

We consider a large number of oil producers, which can be viewed either as wells or from a more macro standpoint as oil companies. The only assumption we make is that there is a sufficiently large number of them and that one can apply simple hypotheses such as that of the continuum (mean field games modeling) and perfect competition (price-taker behavior of agents).
It's important to notice that the number of producers and hence the size of the continuum is fixed in our model. This may seem a very restrictive economic setting because it goes against the classical economic hypothesis of free entry. Though, it is natural in the field we consider since one cannot create an oil field or a well as an entrepreneur creates a new firm. The number of wells is indeed fixed since we do not consider, for now, any exploration process.
Each of our oil producers initially has a reserve that is termed R_0 and that these reserves are distributed among producers according to an (initial) distribution $m(0, \cdot)$. These reserves will of course contribute to production q such that, for any specific agent, we have $dR(t) = -q(t)dt$.

Production choices will be made in order to optimize a profit criterion (the same for all agents) of the following form:

$$Max_{(q(t))_t, T} \int_0^T (p(t)q(t) - C(q(t)))e^{-rt} ds \quad s.t.\ q(t) \geq 0, R(t) \geq 0$$

where:

- C is the production cost function which we will then write as quadratic: $C(q) = \alpha q + \beta \frac{q^2}{2}$.
- the prices $p(t)$ are determined according to a rule that will most often be the supply/demand equilibrium on the market at each moment, demand being given by a function $D(t, p(t))$[1] at instant t and supply naturally

[1] The demand function for oil will take several forms in this text. A first form that will be

given by the total oil production of the producers.

2.1.2 Characterization of the equilibrium

The problem we consider can be completely solved using classical Euler-Lagrange tools. Let's consider the problem faced by a producer with reserve R_0. This problem can be summed up by a Lagrangian (if we omit for now the condition on the positiveness of the production):

$$\mathcal{L} = \int_0^T (p(s)q(s) - C(q(s)))e^{-rs}ds + \lambda(R_0)\left(R_0 - \int_0^T q(s)ds\right)$$

where $\lambda(R_0)$ is the shadow price of the reserve constraint $\int_0^T q(s)ds = R_0$. This shadow price will play an important role in the following characterization of the solution:

Proposition 2.1 (Equilibrium in the deterministic case)**.** *The equilibrium is characterized by the following equations where $t \mapsto p(t)$, $(t, R_0) \mapsto q(t, R_0)$ and $R_0 \mapsto \lambda(R_0)$ are unknown functions and where the levels of initial oil reserves are denoted by R_0.*

$$D(s, p(s)) = \int q(s, R_0)m_0(R_0)dR_0$$

$$q(s, R_0) = \frac{1}{\beta}\left[p(s) - \alpha - \lambda(R_0)e^{rs}\right]_+$$

$$\int_0^\infty q(s, R_0)ds = R_0$$

Remark: The first equation corresponds to the supply/demand equilibrium and defines each $p(s)$. The second equation characterizes the optimal scheme

used in an example below is $D(t, p) = Ee^{\rho t}(\pi_{sub} - p)$ where E is a global wealth factor, where ρ stands for the growth rate of the economy, and where π_{sub} is the price of a substitute that is here exogenous. Other forms will be taken such as a CES (Constant Elasticity of Substitution) demand function: $D(t, p) = Ee^{\rho t}p^{-\sigma}$. This latter form expresses that oil demand grows by σ% when the price decreases by 1%. This form can be generalized to take in account the influence of different energies and substitution between them (see below). A last form will be used to take into account the existence of a price at which oil may be substituted by another good, this price evolving with time. This last form is $D(t, p) = Ee^{\rho t}p^{-\sigma} - \delta$ where a high δ stands for a low price of substitution.

of production where $\lambda(R_0)$ is unknown. The third equation is the intertemporal production constraint, hence the constraint that "defines" the lagrange parameter $\lambda(R_0)$.

Proof:

Let's consider the problem of an oil producer with an oil reserve equal to R_0.

The optimal production levels can be found using the Lagrangian:

$$\mathcal{L} = \int_0^T (p(s)q(s) - C(q(s)))e^{-rs}ds + \lambda(R_0)\left(R_0 - \int_0^T q(s)ds\right)$$

The first order conditions are:

$$\forall s \leq T, \quad p(s) - C'(q(s)) = \lambda(R_0)e^{rs}$$

$$\Rightarrow p(s) = C'(q(s)) + \lambda(R_0)e^{rs}$$

and

$$(p(T)q(T) - C(q(T)))e^{-rT} = \lambda(R_0)q(T)$$

Hence, using our specification for the costs, we get:

$$\forall s \leq T, p(s) - \alpha - \beta q(s) = \lambda(R_0)e^{rs}$$

$$q(T)\left[p(T) - \alpha - \beta q(T) - \lambda(R_0)e^{rT}\right] = 0$$

We deduce from this that $q(T) = 0$ or $T = \infty$.

From the first condition we get that, before the end of the production, $q(s)$ is given by

$$q(s) = \frac{1}{\beta}\left[p(s) - \alpha - \lambda(R_0)e^{rs}\right]$$

Hence, we can simply define the production function for all s by:

$$q(s) = \frac{1}{\beta}\left[p(s) - \alpha - \lambda(R_0)e^{rs}\right]_+$$

In this equation $\lambda(R_0)$ depends on the initial oil stock (or reserve). This lagrangian multiplier is given by the inter-temporal constraint that equalizes the whole stream of production and the initial oil reserve:

$$\int_0^T q(s, R_0)ds = \frac{1}{\beta}\int_0^\infty \left(p(s) - \alpha - \lambda(R_0)e^{rs}\right)_+ ds = R_0$$

Now, we need to find the prices that were left unknown. This simply is given by the demand/supply equality.

$$D(s, p(s)) = \int q(s, R_0)m_0(R_0)dR_0$$

If we compile all these results we get the 3 equations that characterize the equilibrium. □

2.2 Hotelling rent

In the proof of the preceding proposition, we exhibited a relation between prices and marginal cost. For a non-exhaustible resource, because of the pure competitive framework, marginal cost equals price for the optimum. Here there is a rent, called Hotelling rent that makes the price larger than the marginal cost of production on the optimal trajectory. The formula has been given above and is:

$$p(t) = C'(q(t)) + \underbrace{\lambda(R_0)e^{rt}}_{Hotelling\ rent}$$

Some remarks must be made on this rent. First the rent is increasing with time at a constant growth rate r. Second, the rent depends on the Lagrange multiplier. Because $\lambda(R_0)$ is the shadow price of the constraint, the more important the constraint is, the larger $\lambda(R_0)$ is. Hence, $\lambda(\cdot)$ is a decreasing function. This means that the rent is larger for a small producer than for a large producer.

Finally, we presented the rent in a competitive framework but it's important to notice that, in the general case, the rent is a function of the competition intensity. For instance, if one considers a monopoly, the price is usually the marginal cost times a markup. Here, both the marginal cost and the above (competitive) rent have to be multiplied by the usual markup to obtain the price. Hence, the rent of exhaustibility is in fact multiplied by the markup.

Now, if we look at the formula that expresses the evolution of the production, namely $q(s, R_0) = \frac{1}{\beta}\left[p(s) - \alpha - \lambda(R_0)e^{rs}\right]_+$, we see that there are two forces at work. First, if we consider that the price will end up growing, the production will grow with time. This is the first effect at work but the right part of the equation (the rent) pushes the production towards 0 and indeed it must be equal to 0 asymptotically. These two forces can lead to several evolution schemes for the production. Oil production can indeed decrease with time or increase first and then decrease towards 0. The latter case is similar to what is usually termed Hubbert peak, though we are dealing with wells whose reserve is fixed at start.
In what follows, we discuss the evolution on a very specific example.

2.3 An explicit example and the Hubbert Peak

Let's consider a very specific case in which all the wells are of the same size R_0. In other words, we consider a case in which everybody is going to act similarly and hence the derivation of the equilibrium will be made easily. To simplify even more, we are going to consider a simple demand function which is $D(t, p(t)) = Ee^{\rho t}(\pi_{sub} - p(t))$ where π_{sub} models the price of a substitute.

In this context, the supply/demand equilibrium can be written $q(t) = Ee^{\rho t}(\pi_{sub} - p(t))$ and hence we can solve the problem for $(p(t), q(t))$, leaving $\lambda = \lambda(R_0)$ unknown.
The system is (as long as there is production):

$$\begin{cases} q(t) = \frac{1}{\beta}\left(p(t) - \alpha - \lambda e^{rt}\right) \\ q(t) = Ee^{\rho t}(\pi_{sub} - p(t)) \end{cases}$$

$$\implies q(t) = \frac{E}{\beta E + e^{-\rho t}}\left[\pi_{sub} - \alpha - \lambda e^{rt}\right]$$

We have in this example the two effects discussed above. First, $\frac{E}{\beta E+e^{-\rho t}}$ is increasing, because of the positive growth rate ρ (that is here exogenous, for the sake of simplicity). This growth effect induces more and more demand, *ceteris paribus*, and hence this growth process tends to induce an increasing flow of production. Second, the term $\pi_{sub} - \alpha - \lambda e^{rt}$ is decreasing and tends to push the production towards 0. Since $t \mapsto q(t)$ is either decreasing, or increasing and then decreasing, we can obtain a condition to have an inverse-U shape for the production.

Proposition 2.2 (Hubbert peak)**.** *In the preceding context, there is a "Hubbert peak" if and only if* $\lambda < \lambda^*$ *where*

$$\lambda^* = \frac{\rho(\pi_{sub} - \alpha)}{\rho + r(1+\beta E)}$$

Proof:

The condition to have a Hubbert peak in the preceding context is simply $q'(0) > 0$. We can see that:

$$q'(0) = \frac{\rho E}{(1+\beta E)^2}(\pi_{sub} - \alpha - \lambda) - \lambda r \frac{E}{1+\beta E}$$

This condition is positive if and only if $\rho(\pi_{sub} - \alpha) - \rho\lambda > \lambda r(1+\beta E)$ and this gives the result. □

Now, we must recall that λ is a function of the oil reserve ($\lambda = \lambda(R_0)$). Since this function is decreasing, the above formula says that a Hubbert peak will appear if the reserves are high enough. This is clear in this specific model but the phenomenon is the same in general when wells are of different sizes: large oil producers will be characterized by a Hubbert peak whereas small oil producers will rapidly get rid of their oil reserves.
Interestingly, the threshold R_0^* associated to λ^* (*i.e.* $\lambda^* = \lambda(R_0^*)$) varies with the parameters in a clear way:

- R_0^* is decreasing with the growth rate: in more general models, it means that the more growth we have, the more wells will be characterized by a Hubbert peak. Moreover, we see from the formula that a positive growth rate is necessary to have any Hubbert peak. The meaning of the preceding threshold is that it is not sufficient: the economic growth can indeed be absorbed by quantities (the Hubbert peak) or by prices (it is the case after some time).

- R_0^* is decreasing with the price of the substitute: since price may be greater when the price of the substitute is costly, there is a higher incentive to postpone production.
- R_0^* is increasing with the interest rate: a high interest rate simply does not incite to postpone production.

2.4 Computation of an equilibrium

2.4.1 Eductive methods

Now that we have presented the model and solved it in a very special case to understand the different effects at work, we can consider the problem of finding a solution numerically in the general case. To solve this problem let's go back to the characterization of the equilibrium given in Proposition 2.1.

We see that $(t, R_0) \mapsto q(t, R_0)$ only depends on $R_0 \mapsto \lambda(R_0)$ and $t \mapsto p(t)$. Hence we can totally separate the variables t and R_0. More precisely, if we consider an eductive algorithm (eductive algorithms will be used later to solve coupled partial differential equations) we can consider two "guesses" $\lambda(\cdot)$ and $p(\cdot)$ to compute $q(\cdot, \cdot)$ and then update $\lambda(\cdot)$ and $p(\cdot)$ using respectively the constraints $\int_0^\infty q(s, R_0)ds = R_0$ and $D(s, p(s)) = \int q(s, R_0)m_0(R_0)dR_0$.

More precisely, if for a given t, the price $p(t)$ does not verify the supply/demand equilibrium equation, let's say $D(t, p(t)) > \int q(t, R_0)m_0(R_0)dR_0$, then the price has to move (it must go up in the case considered to limit the excess of demand). Hence, we can consider adding a time θ and define a function $(t, \theta) \mapsto p(t, \theta)$ by an initial function (called initial guess) $t \mapsto p(t, 0)$ and

$$\partial_\theta p(t, \theta) = D(t, p(t, \theta)) - \int q(t, R_0, \theta)m_0(R_0)dR_0$$

Similarly, we can update an initial guess for λ and we get the following equation

$$\partial_\theta \lambda(R_0, \theta) = \int_0^\infty q(t, R_0, \theta)dt - R_0$$

where we use the following notation:

$$q(t, R_0, \theta) = \frac{1}{\beta}\left[p(t, \theta) - \alpha - \lambda(R_0, \theta)e^{rt}\right]_+$$

In fact the idea is more general. We can indeed consider a system of integral-differential equations for any two increasing functions Ψ_p and Ψ_λ:

$$\begin{cases} \partial_\theta p(t, \theta) = \Psi_p\left(D(t, p(t, \theta)) - \int q(t, R_0, \theta)m_0(R_0)dR_0\right) \\ \partial_\theta \lambda(R_0, \theta) = \Psi_\lambda\left(\int_0^\infty q(t, R_0, \theta)dt - R_0\right) \\ q(t, R_0, \theta) = \frac{1}{\beta}\left[p(t, \theta) - \alpha - \lambda(R_0, \theta)e^{rt}\right]_+ \end{cases}$$

Considering such functions Ψ_p and Ψ_λ or replacing the variable on which we elucidate (for example, on may want to consider $\partial_\theta \ln(p(t, \theta))$ instead of $\partial_\theta p(t, \theta)$) is important to better control the size of the moves in λ and p. Once a system is chosen and seem to converge in practice, the idea numerically is to obtain the solution for $R_0 \mapsto \lambda(R_0)$ and $t \mapsto p(t)$ (and hence the productions of all oil producers) using the following limits:

$$\lim_{\theta \to +\infty} p(t, \theta) = p(t)$$

$$\lim_{\theta \to +\infty} \lambda(R_0, \theta) = \lambda(R_0)$$

2.4.2 Examples

As a first example[2] we can illustrate the evolution of total oil production in this model where we consider a CES demand function, namely $D(t, p) = Ee^{\rho t}p^{-\sigma}$.

We took the following values for the parameters: the interest rate considered by oil producers is $r = 5\%$, the average growth rate of the world economy

[2] The MATLAB source codes that generate those results are presented in the appendix and may be adapted to other parameters straightforwardly

is $\rho = 2\%$, the initial marginal cost of producing an oil barrel is $\alpha = 10$. We took $\beta = 100$ to model the importance of capacity constraints, $\sigma = 1.2$ because oil is not a highly elastic good and $E = 40$ to obtain meaningful values in the model. The problem is considered over 150 years and the initial distribution of reserves is given by Figure 1.

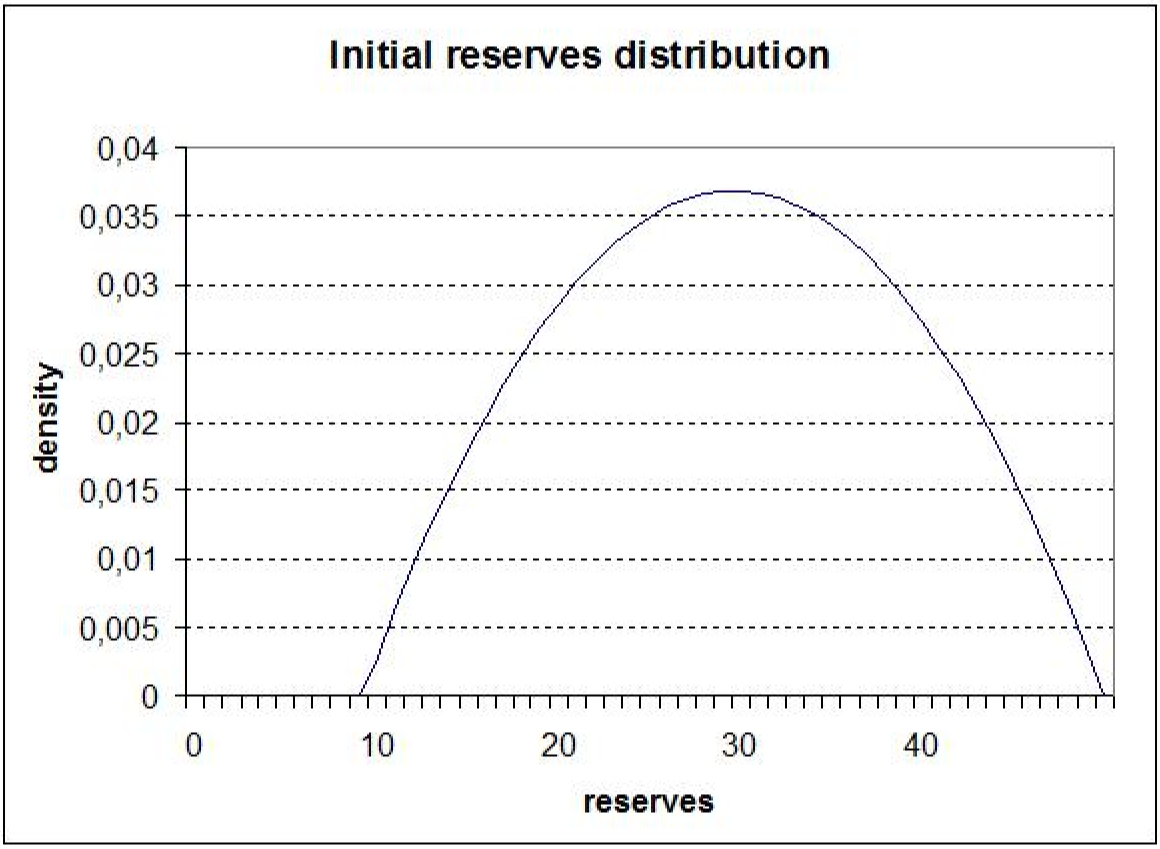

Figure 1: m_0

The evolution of the total production of oil producers is plotted on Figure 2 and the associated evolution of oil prices is on Figure 3 where we only plot the first 50 years to avoid ending up with very large values after too many decades and hence a graph that is unreadable.

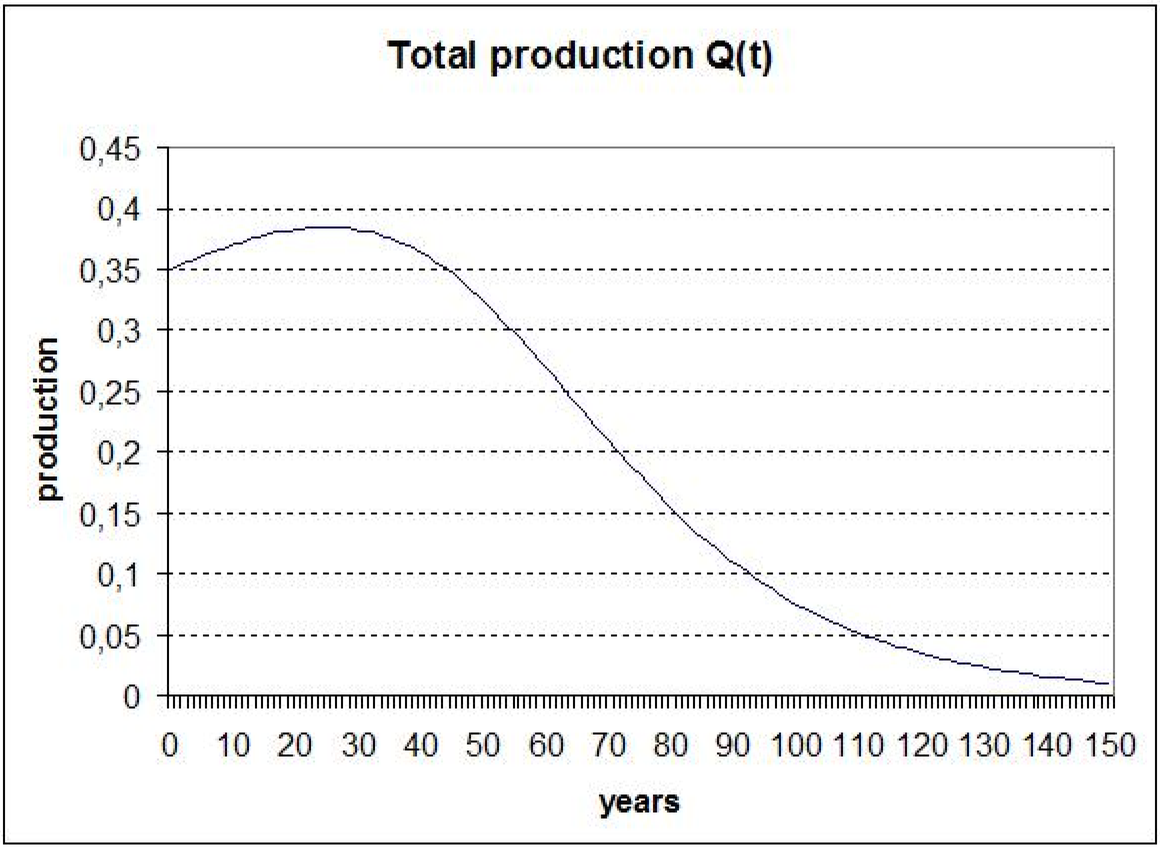

Figure 2: Evolution of the total oil production

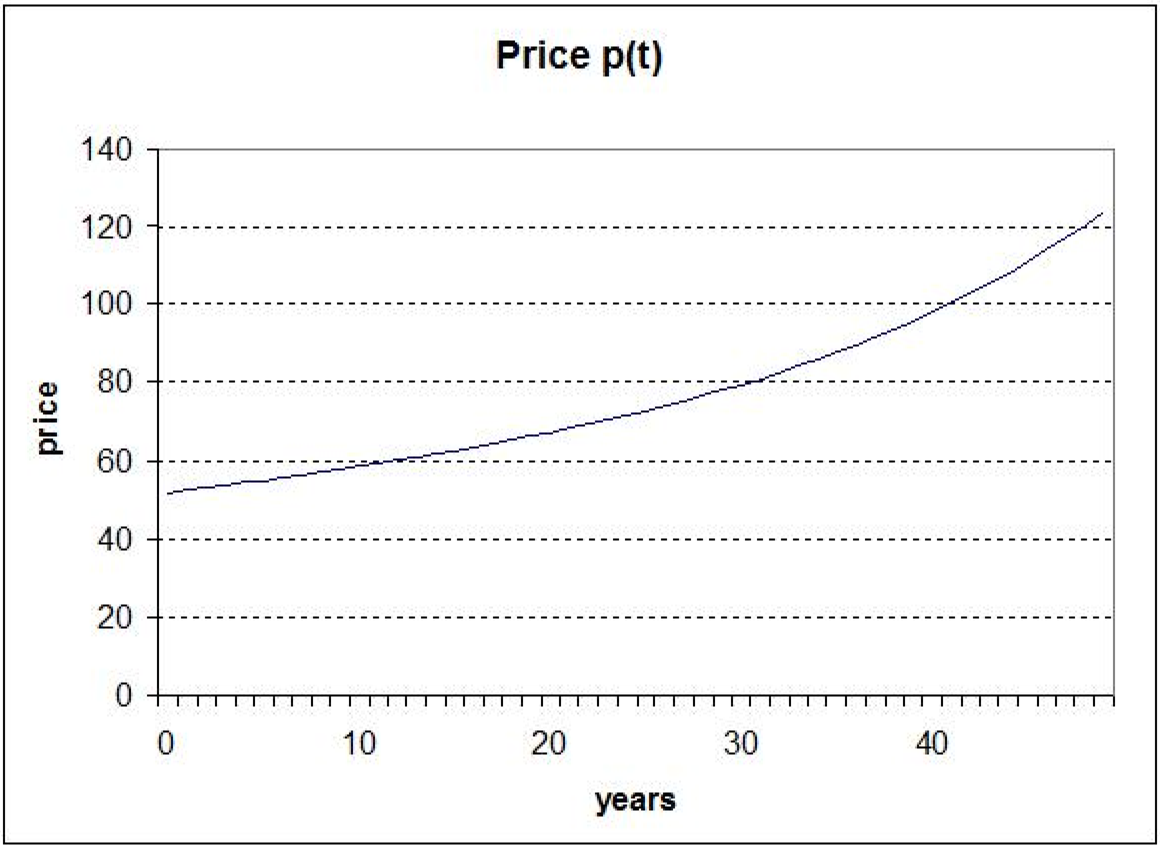

Figure 3: Evolution of prices over 50 years

Now, we can consider another demand function, namely $D(t,p) = Ee^{\rho t}p^{-\sigma} - \delta$ to model the fact that there is a maximum price for oil. This maximum price for oil corresponds to the price at which the demand for oil vanishes and hence, with our settings, the maximum price evolves at constant rate ρ/σ. Leaving unchanged the other parameters, we decided to present an example with $\delta = 0.1$, hence setting today maximum price to \$150. The results for the total production are shown on Figure 4 and for the evolution of oil prices we refer to Figure 5, where we stopped to plot prices when no oil was remaining.

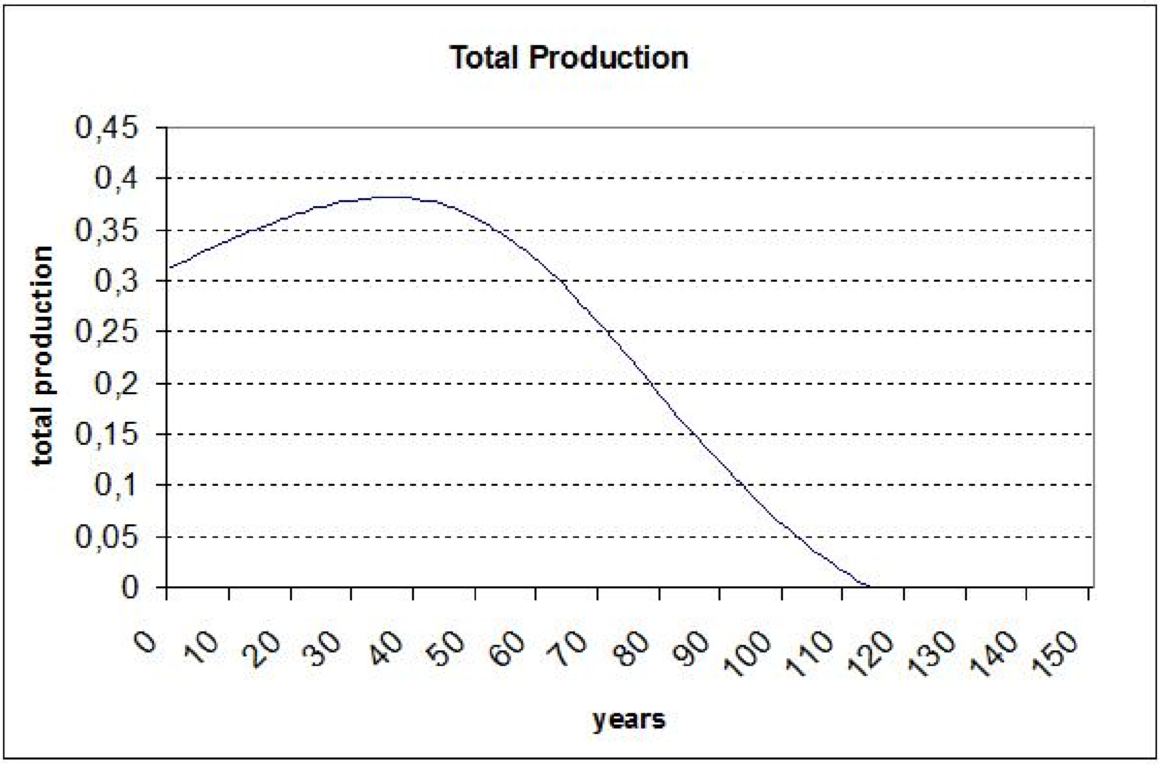

Figure 4: Evolution of the total oil production when $\delta = 0.1$

We see that, because we add a maximum price at which oil cannot be sold, the production is faster and no oil remains in the ground after 115 years. The interpretation is straightforward: since prices cannot grow as high as in the preceding model, there is a less important incentive to wait for high prices and hence the overall production is faster.

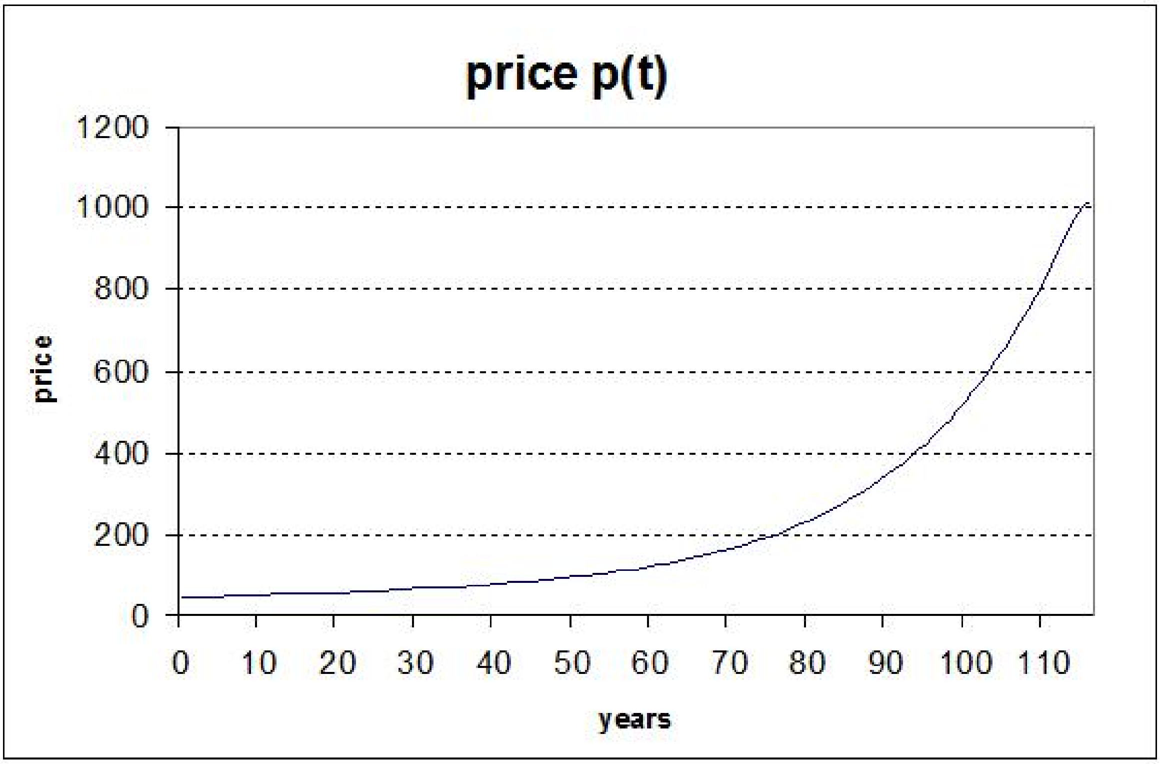

Figure 5: Evolution of prices when $\delta = 0.1$

2.5 Comments on the deterministic model

2.5.1 A short digression on perfect competition and Nash equilibria

Let's consider for the sake of simplicity the second case in which no oil is remaining in the ground after 115 years. Oil prices do not exist after this period since there does not exist anymore an oil market. In fact our oil market has less and less participants as time t get closer and closer to the final period $t = 115$. Hence, if our continuum hypothesis may be valid mathematically, it's in practice difficult to justify why producers still have no market power just before the end of the oil era. One would indeed think that the end of the oil era, in spite of all the uncertainty surrounding it, is better described by a situation of small oligopoly than by a situation of perfect competition. This is one of the problem in modeling the production of exhaustible resources and we highlight this point for future research.

There is another problem, linked to this first one, but that may appear more clearly when there is no maximum price like for the pure CES demand function. Since each producers stop producing at a given date (after time 150 in our case where the graph represented on Figure 2 is an approximation of the actual solution), there is one day when no oil is remaining. However, since there is no maximum price to oil, a producer, taken individually, may keep some oil in the ground and sell it when everybody else has no oil remaining and make an infinite profit. Hence, our equilibria are not always Nash equilibria. This is also linked to the very exhaustible nature of oil. In fact the methodology used above gives an (Nash) equilibrium, with the hypothesis of pure competition, if we consider the period before the end of the oil era. But the very end of oil era is determined endogenously and we see that mathematically, this definition of an equilibrium is rather weak. This caveat will have to be dealt with care when we use a mean field method because the price function have to be defined for all t (see below).

2.5.2 On the methodology

The deterministic model presented above do not use uncommon mathematical methods. Except perhaps our eductive method to solve the problem numerically - eductive methods are largely used to solve mean field games partial differential equations and will be used in the stochastic model that follows -, everything is based on Euler-Lagrange tools.
However, in this deterministic framework, there is no room for externality. How can we model, for instance, the will of producers to produce more rapidly than others to avoid being one of the last actors in oil production? Because of their risk-aversion, producers may indeed want to avoid considering extreme scenarios like nationalizations of oil reserves or price control for example, at the end of the oil era.

The mean field game framework we are now going to present will prove to be well suited for generalizations. Moreover, it will be considered in a stochastic model, where reserves, or more exactly the estimation of reserves, are affected by a brownian motion.

3 The stochastic model - A MFG framework

Although we did not emphasize on this point, the preceding model was a mean field game as any general equilibrium economic model. In the simple deterministic case developed above, the mean field games tools didn't need to be used and classical tools were sufficient. However, when it comes to add noise or to consider externality in the model, the mean field games partial differential equations will be necessary.

3.1 The mean field games partial differential equations

To start writing the equations, let's introduce randomness in the model. Instead of considering a deterministic evolution of the reserve with production (namely $dR(t) = -q(t)dt$), we may think that the estimation of reserves is not perfect or that reserves are indeed random: a well can turn out to be more complicated to exploit than expected or on the contrary additional oil can be found (though we do not explicitly model research of new oil reserves). To model these ideas we suppose that each producer i is affected by a idiosyncratic brownian motion and hence:

$$dR^i(t) = -q^i(t)dt + \nu R^i(t)dW_t^i$$

where ν measures the intensity of the noise. This intensity (or volatility) is supposed to be the same for all producers but the noise is proportional to the oil reserve.

Production choices will be made in order to optimize a profit criterion similar to the one used before:

$$Max_{(q(t))_t}\mathbb{E}\left[\int_0^\infty (p(t)q(t) - C(q(t)))e^{-rt}dt\right] \quad s.t.\ q(t) \geq 0, R(t) \geq 0$$

To this criterion, we associate a Bellman function $u(t, R)$ namely:

$$u(t,R) = Max_{(q(s))_{s\geq t}, q\geq 0}\mathbb{E}_t\left[\int_t^{\infty}(p(s)q(s) - C(q(s)))e^{-r(s-t)}ds\right]$$

$$s.t.\ dR(s) = -q(s)ds + \nu R(s)dW_s, \quad R(t) = R, \quad \forall s \geq t, R(s) \geq 0$$

The Bellman function verifies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$(HJB) \quad \partial_t u(t,R) + \frac{\nu^2}{2}R^2\partial^2_{RR}u(t,R) - ru(t,R)$$
$$+ \max_{q\geq 0}\left(p(t)q - C(q) - q\partial_R u(t,R)\right) = 0$$

The Hamiltonian of this problem is $\max_{q\geq 0}(p(t)q - C(q) - q\partial_R u(t,R))$. If we keep considering the quadratic cost function that has been used so far, then the optimal control is given by:

$$q^*(t,R) = \left[\frac{p(t) - \alpha - \partial_R u(t,R)}{\beta}\right]_+$$

where $q^*(t,R)$ represents the instantaneous production at time t of a producer with an oil reserve R at time t. It's important to notice that R designates the reserve at time t and not the initial reserve as in the deterministic model presented above.

The Hamilton-Jacobi-Bellman equation can be rewritten using the optimal production:

$$\partial_t u(t,R) + \frac{\nu^2}{2}R^2\partial^2_{RR}u(t,R) - ru(t,R) + \frac{1}{2\beta}\left[(p(t) - \alpha - \partial_R u(t,R))_+\right]^2 = 0$$

Now, let's denote $m(t,R)$ the distribution of oil reserves at time t. This distribution is initially given by a function $m_0(\cdot)$ and then transported by the optimal production decisions of the agents $q^*(t,R)$.

The transport equation is:

$$(Kolmogorov) \qquad \partial_t m(t,R) + \partial_R(-q^*(t,R)m(t,R)) = \frac{\nu^2}{2}\partial^2_{RR}\left[R^2 m(t,R)\right]$$

with $m(0,\cdot)$ given by $m_0(\cdot)$.

These two equations are the two classical partial differential equation of the mean field games theory. The Hamilton-Jacobi-Bellman equation is a backward equation whereas the transport equation is a forward one. The link between the two is the double coupling. As usual in the mean field games framework, these equations are indeed coupled but here the double coupling is not obvious at first sight. Firstly, m depends on u through the optimal production function q^* and secondly, the MFG coupling is characterized by prices. In the HJB equation, the Hamiltonian depends on the price function $t \mapsto p(t)$ and we have to recall that prices are fixed by a global (as opposed to individual) equilibrium between supply and demand. Since supply depends on the production of all producers, it depends on the distribution of reserves amongst producers: m.

Mathematically, $p(t)$ is fixed so that supply and demand are equal and hence:

$$p(t) = D(t,\cdot)^{-1}\left(\underbrace{-\frac{d}{dt}\int Rm(t,R)dR}_{supply}\right)$$

Hence, the coupling is indeed double: m depends on u and u depends on m. The type of interdependence involved here is not that common since the coupling happens into the Hamiltonian and not as a second member of the Hamilton-Jacobi-Bellman equation.

If we want to sum up and rewrite the equations to focus on the interdependence, we may write the following Proposition:

Proposition 3.1 (Mean field games PDEs)**.** *The two partial differential equa-*

tions associated to our oil problem can be written:

$$\partial_t u(t,R) + \frac{\nu^2}{2} R^2 \partial^2_{RR} u(t,R) - ru(t,R)$$

$$+\frac{1}{2\beta}\left[\left(D(t,\cdot)^{-1}\left(-\frac{d}{dt}\int Rm(t,R)dR\right) - \alpha - \partial_R u(t,R)\right)_+\right]^2 = 0$$

and

$$\partial_t m(t,R) + \partial_R\left(-\left[\frac{D(t,\cdot)^{-1}\left(-\frac{d}{dt}\int Rm(t,R)dR\right) - \alpha - \partial_R u(t,R)}{\beta}\right]_+ m(t,R)\right)$$

$$= \frac{\nu^2}{2}\partial^2_{RR}\left(R^2 m(t,R)\right)$$

3.2 Analysis of the model

So far, we have slightly modified the initial model since we basically added noise to the model. However, the first Euler-Lagrange approach and the approach we have just presented are really different and the mean field game approach will be shown to shed light on the fact that our model corresponds to a very particular case, namely general equilibrium, and that the model can be generalized with only few efforts.

3.2.1 Hotelling rent

Before focusing on this aspect, let's go back to the Hotelling rent exhibited in the first model. In the deterministic model, the optimal production was given by an expression of the form

$$q^*(t) = \frac{1}{\beta}\left[p(t) - \alpha - \lambda e^{rt}\right]_+$$

Here, we have a similar expression, namely

$$q^*(t,R) = \left[\frac{p(t) - \alpha - \partial_R u(t,R)}{\beta}\right]_+$$

and hence we can make a parallel between the Bellman function u and the Hotelling rent.
The Hotelling rent is indeed in general the gradient of the Bellman function:

$$Hotelling\ rent\ =\ \partial_R u(t, R)$$

.

This is an interesting reinterpretation of the Hotelling rent and a generalization that will be valid in most of the models we may build on this basic stochastic model.

3.2.2 Prices as a function of the distribution m

In the equations of Proposition 3.1, we see that everything can be modeled (though it is not the most intuitive way) using two functions: m and u. Hence, and we have seen that before, prices and quantities are functions of u and m. The fact that we only take account of a specific functional of m, here $p(t)[m] = D(t,\cdot)^{-1}\left(-\frac{d}{dt}\int Rm(t,R)dR\right)$, prove that our model is specific to a world where the entire analysis goes through prices: a world without any externality. In fact, the functional of m can be changed to model externality and this will be done in one of the next sections.
This remark is not specific to oil production. In fact the general equilibrium theory is a particular case of mean field game and mean field games might be used to model many situation involving externality or, more generally, situations where the optimization criterion is not purely economic but rather social, in a very large sense.

Before presenting extensions of our model, let's focus on the methods to solve the two coupled partial differential equations. Because of the forward/backward structure, there is indeed no obvious and straightforward way to solve the equations.

3.3 Numerical resolution

We are going to present two approaches to solve the problem in the stochastic case. The first one will be a pure PDE method that does not take into account the optimal control dimension of the problem. This first naive method will give solutions to the problem that can be improved a lot using a more complex method inspired from the underlying stochastic control problem. This second method is the one we are going to focus on and the source codes are presented in the appendix.

3.3.1 A naive approach

The two partial differential equations presented above may seem hard to solve numerically. However, we can extend the methodology used in the deterministic case. First of all, let's recall that the two variables considered in the deterministic case were λ and p. Since we noticed that now the Hotelling rent is simply $\partial_R u(t,R)$, we may take u and p as the two variables on which to consider an algorithm.

Before building an algorithm that tries to find a fixed point (u,p), we must transform the problem to take account of the form of the noise. Since the noise is proportional to the amount of reserves, we consider the change of variable $x = \ln(R)$ as often when the noise is a geometrical brownian motion. If we denote $\tilde{m}(t,x) = m(t,e^x)$ and $\tilde{u}(t,x) = u(t,e^x)$ then the partial differential equations become:

$$(HJB_x) \quad \left(\partial_t + \frac{\nu^2}{2}\partial^2_{xx}\right)\tilde{u} - r\tilde{u} - \frac{\nu^2}{2}\partial_x\tilde{u} + \frac{1}{2\beta}\left[\left(p(t) - \alpha - e^{-x}\partial_x\tilde{u}\right)_+\right]^2 = 0$$

where

$$p(t) = D(t,\cdot)^{-1}\left(-\frac{d}{dt}\int e^{2x}\tilde{m}(t,x)dx\right)$$

and

$$(K_x) \quad \left(\partial_t - \frac{\nu^2}{2}\partial^2_{xx}\right)\tilde{m} + e^{-x}\partial_x(-\tilde{q}^*\tilde{m}) = \frac{\nu^2}{2}\left[2\tilde{m} + 3\partial_x\tilde{m}\right]$$

where

$$\tilde{q}^*(t,x) = \left[\frac{p(t) - \alpha - e^{-x}\partial_x \tilde{u}(t,x)}{\beta}\right]_+$$

This way to write the equations with heat operators highlights the forward/backward structure and invite to use these heat operators in the numerical resolution. Our first naive method follows the sequence of steps described below:

1. Take for the initial guesses of $\tilde{u}$ and p the values obtained in the deterministic case, that is considering $\nu = 0$, and taking the preceding methodology. If p can be found directly with our preceding method, u and hence $\tilde{u}$ has to be built from the solution for the production and for the Lagrange multiplier. This is quite straightforward.
2. From $\tilde{u}$ and p, calculate $\tilde{q}^*$
3. From $\tilde{q}^*$ solve the (K_x) equation and obtain a distribution $\tilde{m}$.
4. From $\tilde{m}$, compute a new estimate for $\tilde{p}$.
5. Obtain the new guess for $\tilde{p}$ using a linear combination of the new estimate and the former guess.
6. Solve the (HJB_x) to compute a new estimate for $\tilde{u}$.
7. Update the new estimate for $\tilde{u}$ using a linear combination of the new estimate and the former guess.
8. Go to step 2 until "convergence" is obtained.

Steps 3 and 6 deserved to be detailed since border conditions are not a priori defined. First, if R goes from 0 to $+\infty$, x can take any real value and we must consider a domain $[x_{min}, x_{max}]$ where the bounds have to be chosen considering $\tilde{m}(0,\cdot)$. For the border $x = x_{min}$, we set natural conditions which are $\tilde{u}(t, x_{min}) = 0$ and $\tilde{m}(t, x_{min}) = 0$. These conditions are natural since x_{min} corresponds to the smallest possible level of oil reserve. For x_{max} we decided to impose Neumann condition. This is quite arbitrary but practical. For the initial and final conditions, there is nothing to be said about $\tilde{m}$ because the initial distribution is given but we must impose a condition for a maximum T. After time T we suppose that any remaining oil cannot be sold and hence we have a final condition for the Bellman function that is $\tilde{u}(T,\cdot) = 0$.

This time T has to be chosen using the solution of the problem when $\nu = 0$ to be sure to avoid an important artificial excess of production in the solution.

An example of solution (we represent as before the evolution of the total production) is represented on Figure 6 where the demand function is the same CES function as before.

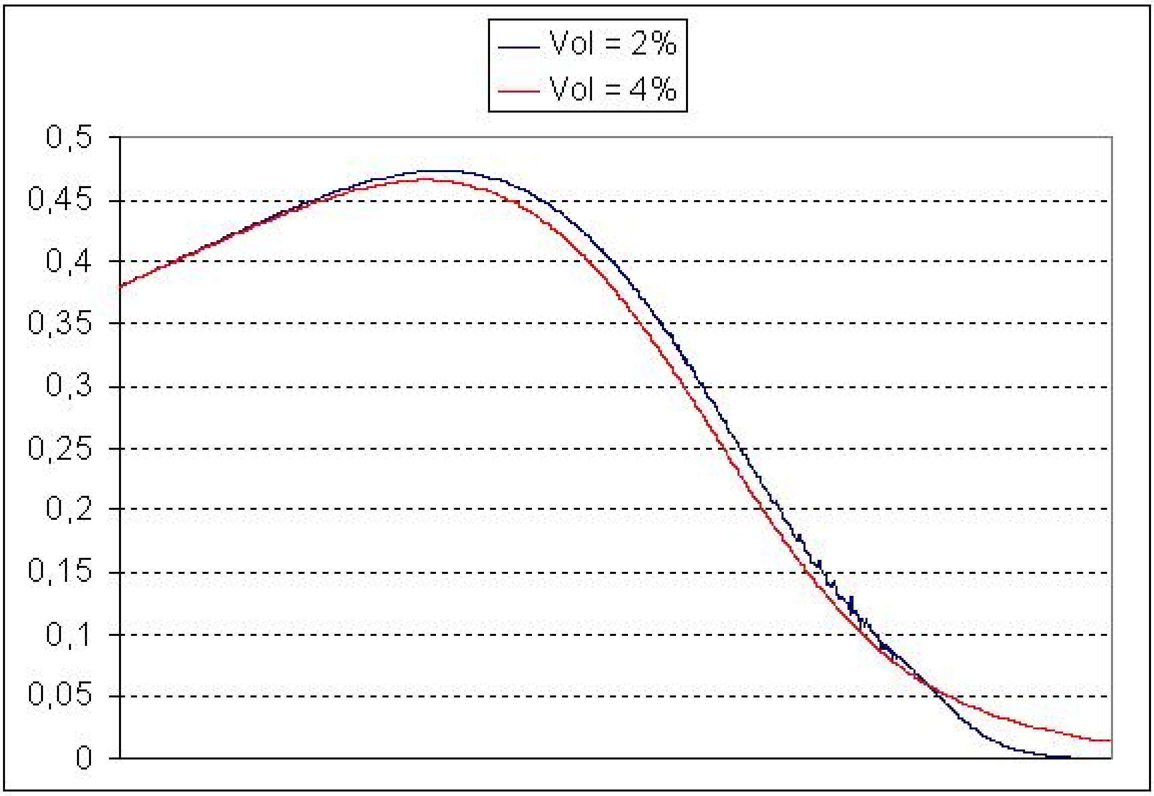

Figure 6: $r = 10\%, \rho = 2\%, \alpha = 10, \beta = 100, \sigma = 1.2, E = 40$

It's important to notice that the curves cannot be obtained directly and that we smoothed the tails of the curves. There is a deep reason for the difficulty to obtain precise tails with a naive finite differences method: a reason that is linked to the exhaustibility of the resource. Because oil is exhaustible, the distribution m is transported towards a distribution that puts more and more weight on small values. But we know that this kind of evolution towards a distribution that resembles a Dirac distribution is very hard to obtain precisely (think of solving a forward heat equation starting from a Dirac distribution from 0 to a given time T and then solving the associated backward equation from time T to time 0: if the schemes are not exactly the same then convergence toward the initial Dirac distribution will never be obtain and we know that oscillations appear).

3.3.2 A more effective methodology

The preceding approach was naive in the sense that all the information from the problem was not used in the methodology employed. We did not considered in fact the underlying stochastic control problem and solved the PDEs using finite differences (though, we used an intuition based on the economic dimension of the model since we tried to find a fixed point in prices, or more exactly in (u, p)).

The method we now present takes account of the underlying control problem and works as follows.

First, because of the economic intuition, we are trying to find a fixed point in p and hence we are considering an eductive method to reach a fixed point p. As always, it's important to define a first guess for p and we are going to use a continuation method for that purpose. An interesting thing to notice is in fact that solving the problem for a demand function that imposes a small maximum price to oil is simpler than solving the problem (*i.e.* finding a fixed point p) for a demand function that imposes a high, if any, maximum price. Hence, if we consider the demand functions $D(t, p) = Ee^{\rho t}p^{-\sigma} - \delta$, the higher the δ, the easier the problem of finding a fixed point t. Intuitively, when δ is high, a fixed point $t \mapsto p(t)$ lies in a relatively "small" space and is then simpler to find than a fixed point for the same problem with a small δ.

Another interpretation of this δ is linked to an exogenous source of production. We can indeed consider that the real demand function is the pure CES demand function $Ee^{\rho t}p^{-\sigma}$ and that δ is produced exogenously, independently of prices. With this interpretation, it is perhaps simpler to understand why a high δ means a problem that is simpler to solve. The higher the δ, the less important the production of the oil producers continuum is relatively to the total oil production (including δ). Hence, with a high δ, prices are not going to move a lot between two steps of the algorithm (we will see just below the algorithm to go from a step to another).
With these remarks, we can start with a high δ, solve the problem and take the solution found as an initial guess for a slightly smaller δ. Progressively, we will find a solution for $\delta = 0.1$ which is our reference case.

Now, for a given δ, we need to describe how to go from a function p to another and hence converge toward the equilibrium p for the given δ.
For this purpose, we are going to consider a price function p and solve the Hamilton-Jacobi-Bellman equation using the underlying optimal control problem. Once the (HJB) is solved and the optimal production function $(t, R) \mapsto q(t, R)$ is determined, we can solve the Kolmogorov equation and hence get the total production $\int q(t, R)m(t, R)dR$ at each date t and derive the associated price using the demand function. New prices will then be a combination of the former prices and of these new prices.

But, we need to explain how to solve the Hamilton-Jacobi-Bellman and the Kolmogorov equations, respectively for a fixed $t \mapsto p(t)$ and for a fixed $(t, R) \mapsto q(t, R)$.
In fact we are going to separate the equations into two parts: the part corresponding to a pure optimal control problem and the part corresponding to the randomness (i.e. the Laplace operator part).
To be more precise, we are going to solve the (HJB) equation using discrete optimal control on a grid of $[0, T] \times [0, R_{max}]$, backward in t. For each t on this grid, we are going to consider for each R the optimal production $q(t, R)$ on the grid as if there were no noise (we find the optimum on the grid but we use a parabolic interpolation to be more precise and avoid being stuck at

a specific node). Once this is done for each R and a given t, we consider the Laplace operator and apply to u a convolution by the appropriate gaussian kernel (depending on the node R, on the time interval between two t's in the grid, and on the volatility ν of the noise). For the transport equation, we use the same distinction between a pure deterministic transport that is obtain through $q(t, R)$ and the stochastic part that is summed up in the Laplace operator.
Importantly, the use of gaussian convolution to replace Laplace operators allows us to consider large time intervals on the grid and hence improve the speed of our algorithm.

Some examples (the MATLAB source codes to obtain these results are presented in the appendix) are shown below where the parameters are identical to those of the second example developed in the deterministic case.

First, let's start consider the mean field games methodology without any noise ($\nu = 0$) to compare the 2 methods. The total production obtained is represented on Figure 7.
The solution is the same as before, the only tiny differences being explained by a difference in the discretization of m_0 that changes a little bit the total amount of oil in the ground.

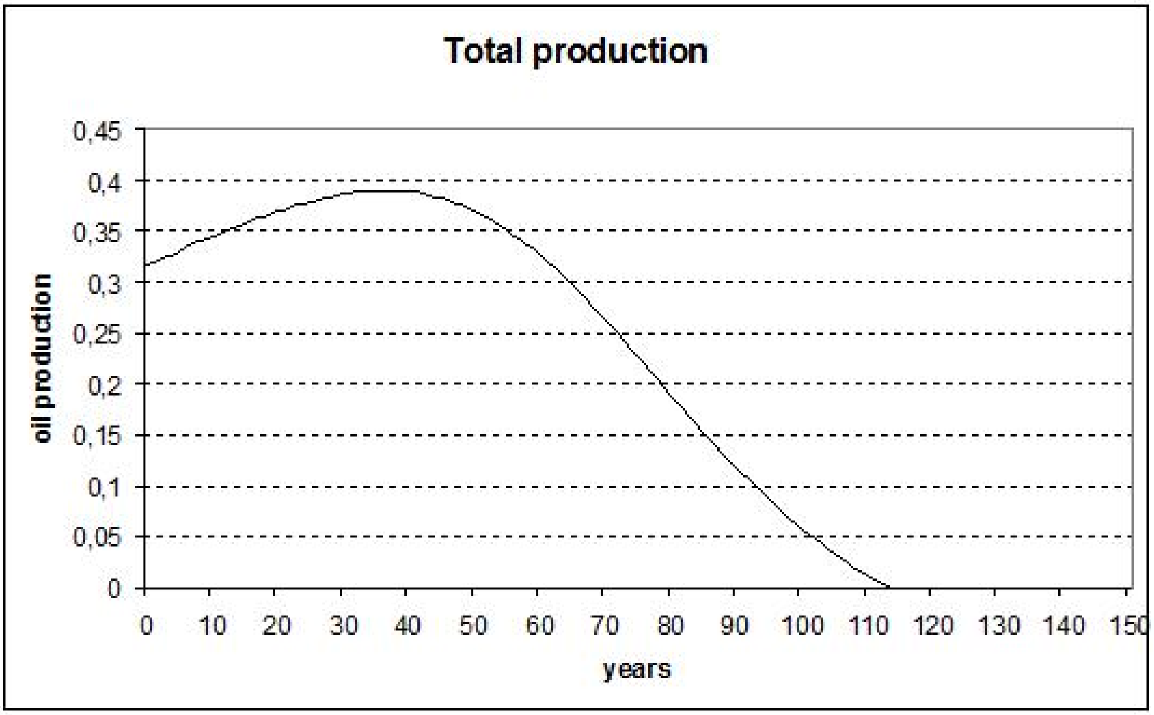

Figure 7: Evolution of the total oil production, $\nu = 0$

Now, it's interesting to consider a problem that could not be solved using Euler-Lagrange tools, namely for $\nu > 0$.

Keeping the same parameters ($r = 5\%, \rho = 2\%, \alpha = 10, \beta = 100, \sigma = 1.2, E = 40, \delta = 0.1$ and m_0 as before) we consider the problem for $\nu = 2\%$ and $\nu = 5\%$. The total production evolution and the evolution of oil prices are given by Figure 8 and Figure 9.
The Bellman functions and the distribution m can also be plotted for several date t (Figure 10 and Figure 11).

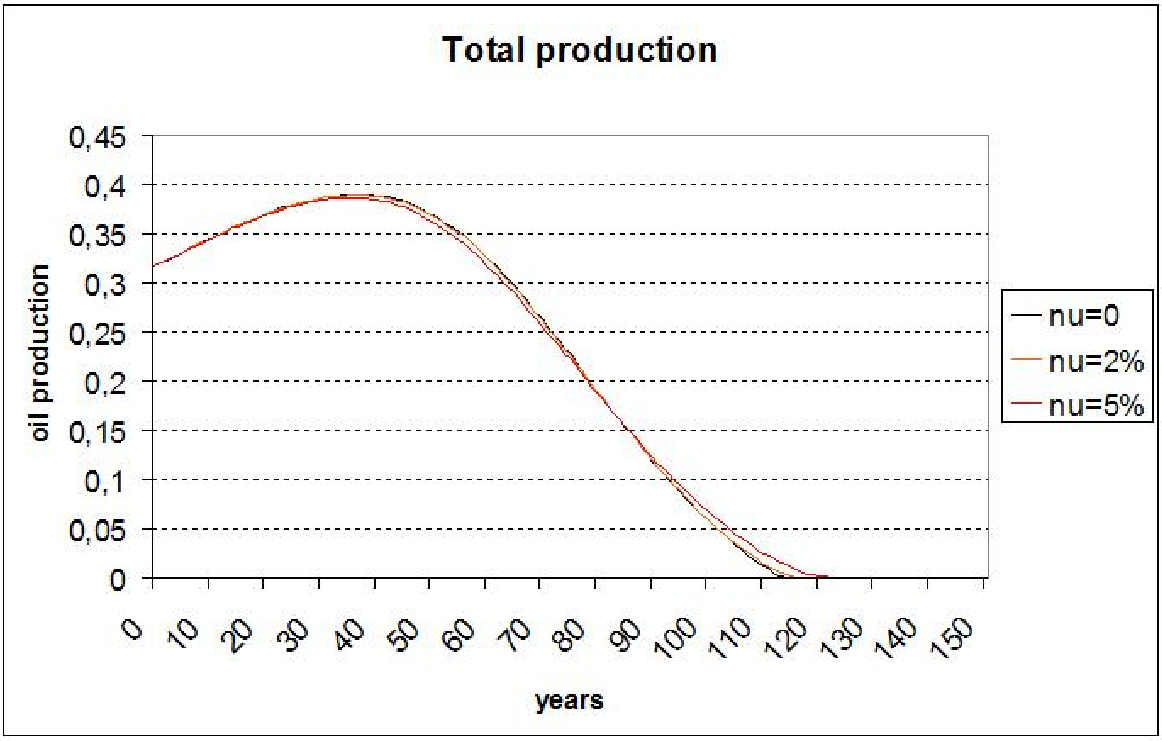

Figure 8: Evolution of the total oil production

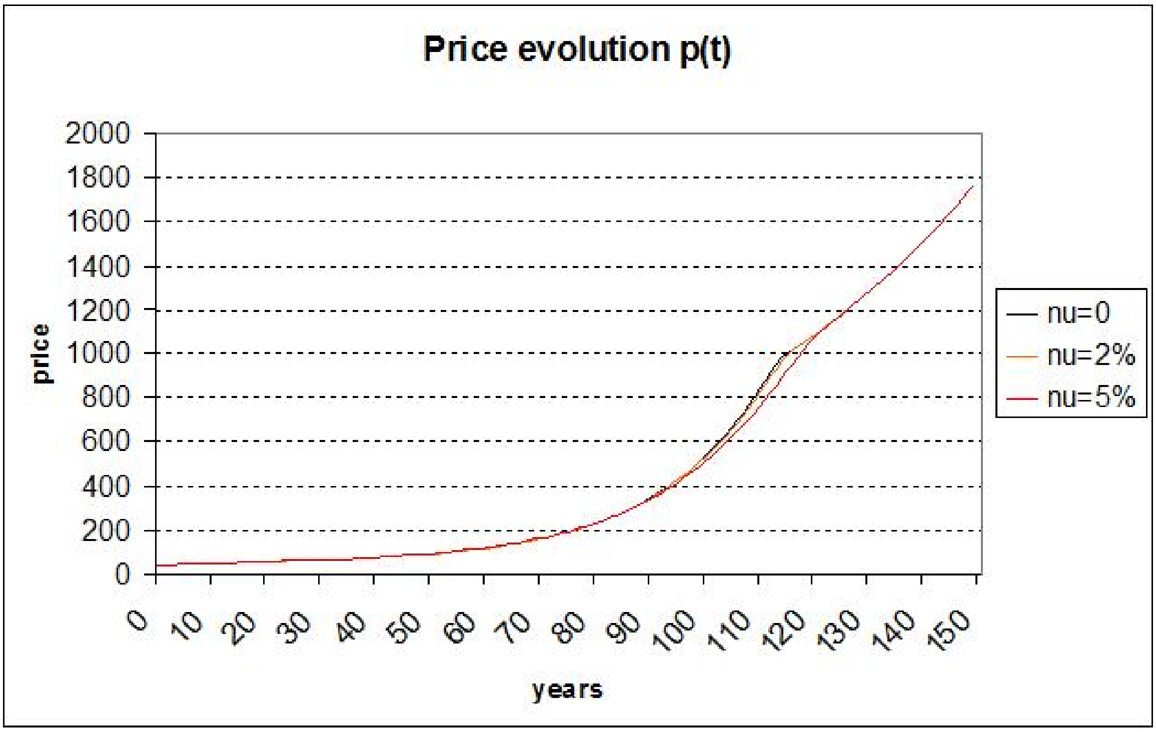

Figure 9: Evolution of prices

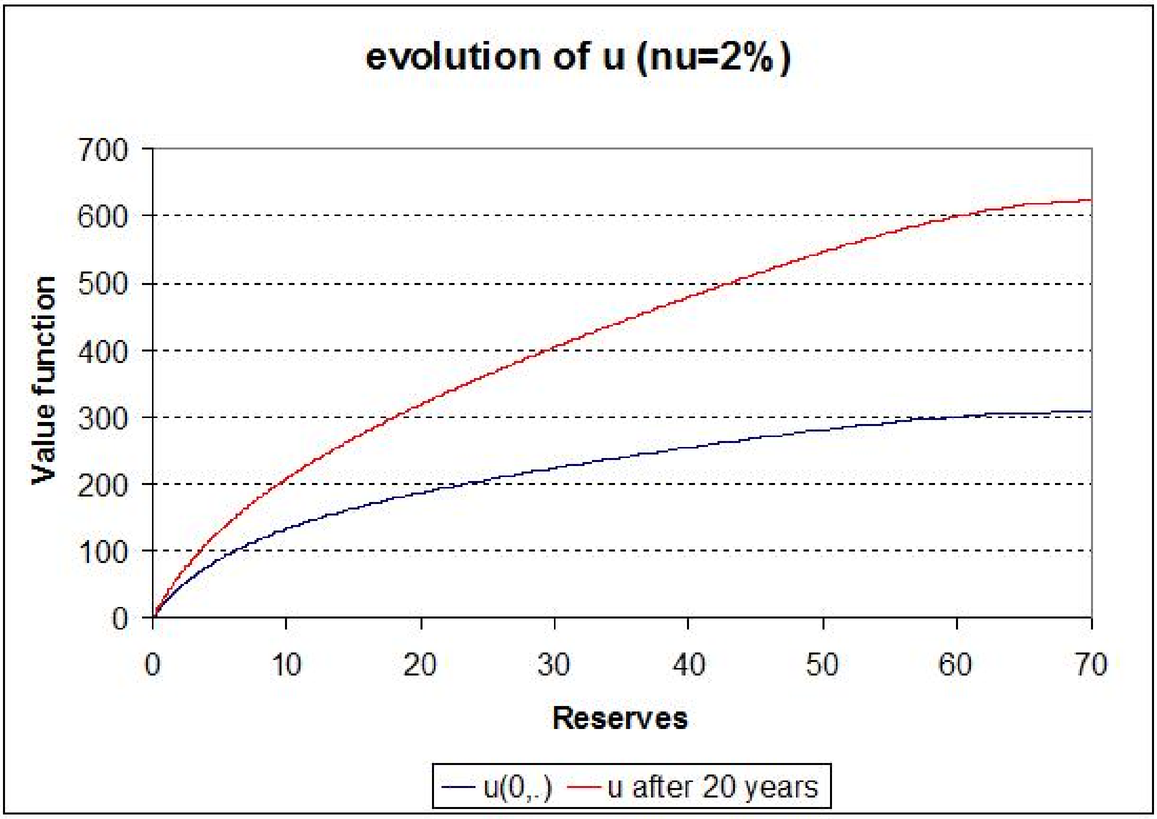

Figure 10: Evolution of the Bellman function for $\nu = 2\%$

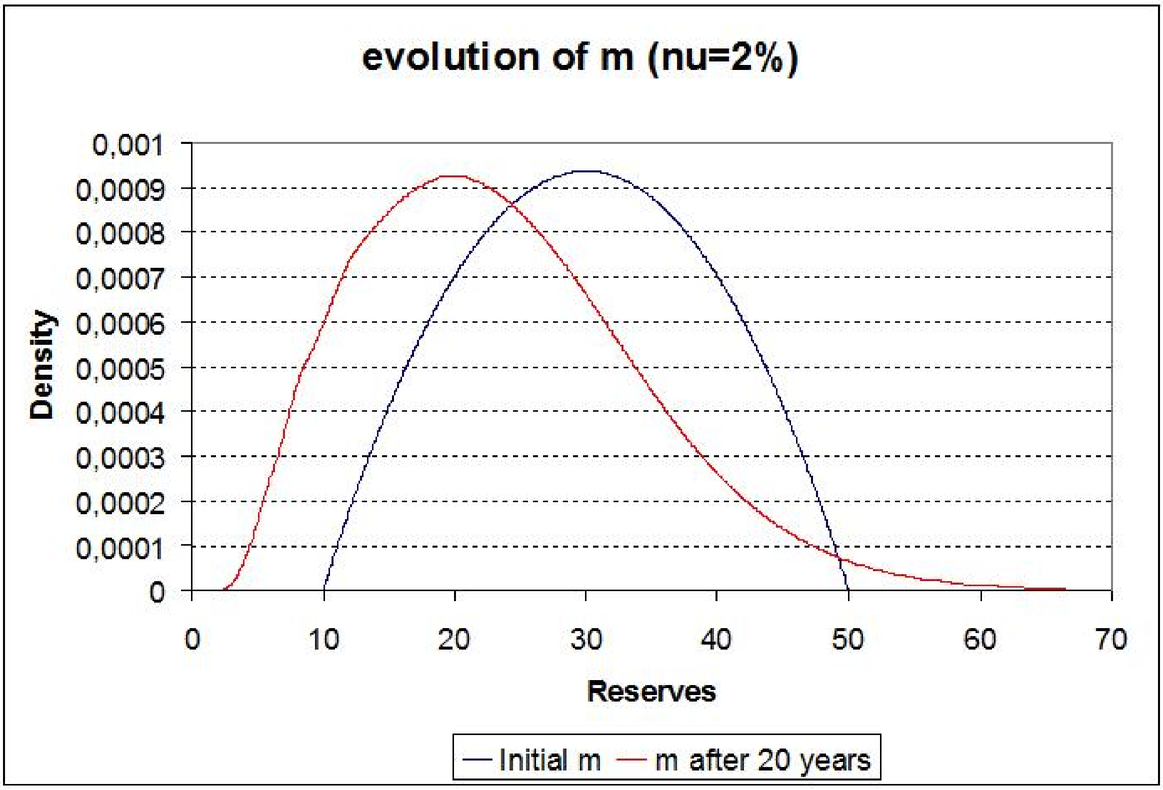

Figure 11: Evolution of the distribution m of reserves for $\nu = 2\%$

3.3.3 Concluding comments on numerical methods

Starting from a naive approach that led to imperfect results, we designed an algorithm aimed at solving the coupled PDEs more precisely using several technical tools: continuation methods, optimization based on the underlying control problem, convolution product to replace Laplace operator and make the algorithm faster.

The algorithm we developed allowed us to consider noise in our model but the impact of noise is not the purpose of this text[3]. The tools we provide are in fact quite general and allow to consider far more complex functions of m than those used in the special case of pure and perfect competition in which producers maximize their inter-temporal profits. This is the purpose of the following section where we use, up to some specific changes, the same algorithm as above, to solve a similar problem with a different optimization criterion, agents being henceforth not only profit-maximizing.

[3]The impact of noise is in fact not clear and mixes risk-aversion effect with price effects that may lead to faster or slower production depending on the shape of the distribution m_0.

4 Generalization and Externality

4.1 Introduction

We have seen before that our model involves a specific function of m (prices in fact) that corresponds to an equilibrium in which all the information is given by prices. We will show in what follows that other aspects (either externality effects or considerations that are not linked to profit) can be modeled using an additional function of m.

As said by Gary Becker during his Nobel lecture (see [2]), economic rationality is not bounded to the maximization of profit. Rationality simply means maximization of a criterion chosen by individuals. For oil producers, there may be other criterium than inter-temporal profit. If, as in our instance, oil producers are interested in their rank in the distribution of reserves, then, their criterium might be a linear combination of inter-temporal profit and rank. Here, we do not question the reason for choosing one or another criterion, we just want to show that the mean field game PDEs allow for more complex criterium than Euler-Lagrange tools.

4.2 Producing before one's peers

4.2.1 The model

The idea we want to introduce is based on the producers' willingness to avoid being one of the last oil producers. This idea is based on risk-aversion, where the risk considered is inherently linked to the end of the oil era. Since prices will increase more and more, it can seem weird to be willing to produce fast and get rid of one's reserve before others. However, this increase in price may be counterbalanced by another effect which is the risk of price control or nationalization because oil is near to exhaustion and still highly strategic. Although we do not directly model this risk, because any modeling will be too specific, we think that a good approach is to model the producers' willingness to get rid of one's reserve before others.
To do that let's consider the ranking function $R \mapsto \int_0^R m(t, \phi) d\phi$. This rank-

ing function varies between 0 and 1. If a producer's ranking is near 0, it means that this producer has only a small oil reserve compared to his peers whereas a ranking near 1 indicates a large producer.
This ranking function is a function of m and can therefore be treated very easily. It's important to notice that this kind of functions of m is archetypal of mean field games: it's a mean field because no one can have, at least individually, an influence on the ranking function but the ranking is a function of everybody's behavior.

Now, let's change the criterion of the preceding model and write the associated Hamilton-Jacobi-Bellman equation.

The optimization criterion we consider will be of the following form, where H is an increasing function to model the willingness to have a relatively small reserve, in comparison to others:

$$Max_{(q(t))_t}\mathbb{E}\left[\int_0^\infty \left(p(t)q(t) - C(q(t)) - H\left(\int_0^{R(t)} m(t,\phi)d\phi\right)\right) e^{-rt}dt\right]$$

$$s.t.\ q(t) \geq 0, R(t) \geq 0$$

where $R(t)$ has the same dynamics as before:

$$dR(t) = -q(t)dt + \nu R(t)dW_t$$

In front of such a criterion, only the mean field games partial differential equations can lead to a solution.

Whereas the Kolmogorov equation is unchanged, the Hamilton-Jacobi-Bellman equation associated to the criterion is simply modified by the adding of the ranking term:

$$(HJB_{rank}) \quad \partial_t u(t,R) + \frac{\nu^2}{2}R^2\partial^2_{RR}u(t,R) - ru(t,R) - H\left(\int_0^R m(t,\phi)d\phi\right)$$

$$+\frac{1}{2\beta}\left[\left(D(t,\cdot)^{-1}\left(-\frac{d}{dt}\int Rm(t,R)dR\right)-\alpha-\partial_R u(t,R)\right)_+\right]^2=0$$

We will see in what follows that solving this equation numerically (coupled with the Kolmogorov equation for the dynamics of m) is relatively easy, once we know how to solve the coupled equations (HJB) and (Kolmogorov) without any ranking effect.

4.2.2 Numerical resolution

To solve (HJB_{rank}) and (Kolmogorov), we are going to use, *mutatis mutandis*, the same algorithm as before. Basically, we are going to use the same continuation method as before and try to find a fixed point in p. However, to solve the Hamilton-Jacobi-Bellman at each step, we need to use a guess for m. Hence, the first difference between our two algorithms is that, at each step, in addition to p we plug the distribution m calculated in the preceding step. This can be done at each continuation step apart from the first one and we consider the usual algorithm without ranking effect for the first continuation step, in order to have a relevant first guess for m.

For numerical purposes, we need to choose a function H and we considered the simple example $H(z) = \epsilon z$. To choose an appropriate range of ϵ's, we must have an idea of the profit made at equilibrium by producers. If we consider the case developed above in this text with $\nu = 2\%$, the profit of a representative agent evolves as depicted on Figure 12.

Thus, we decided to consider two values for ϵ. The first one is $\epsilon = 10$ and corresponds to a case where the ranking effect is relatively important in the criterion but not primary. Another case is considered where $\epsilon = 50$ and in this case, the ranking effect is primordial: get rid of one's reserve before others is relatively more important than making optimal profit. The quantity produced are represented on Figure 13.

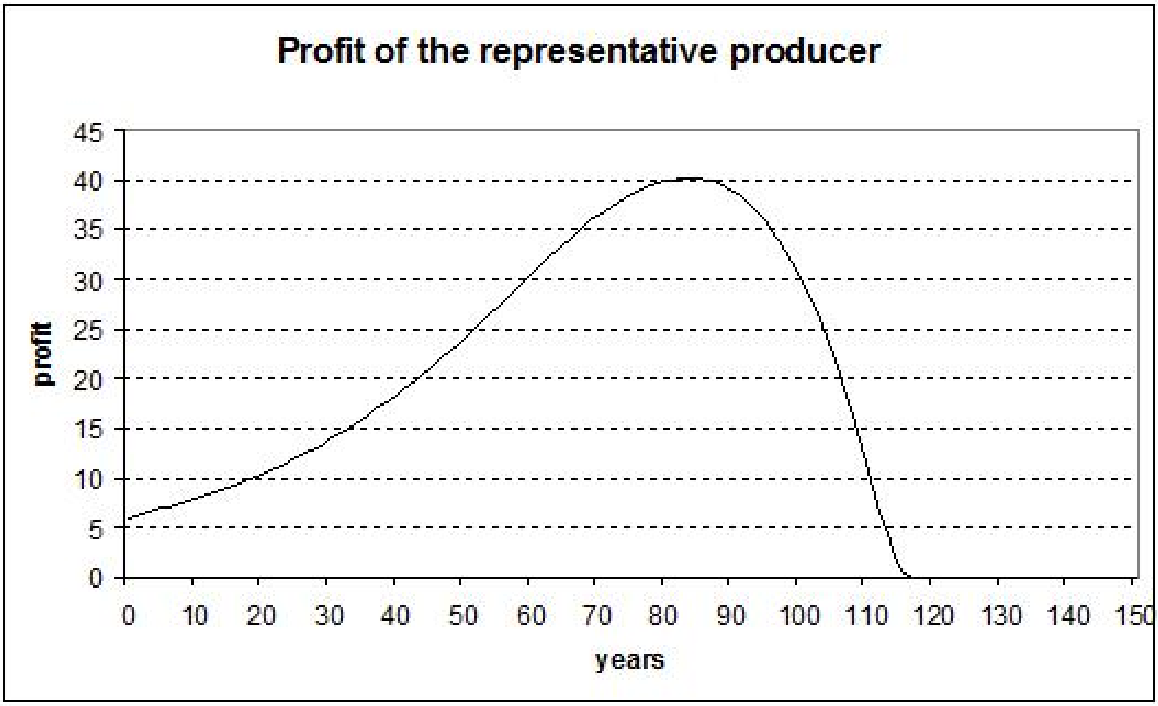

Figure 12: Profit of the average oil producer with $r = 5\%, \rho = 2\%, \alpha = 10, \beta = 100, \sigma = 1.2, \nu = 2\%$

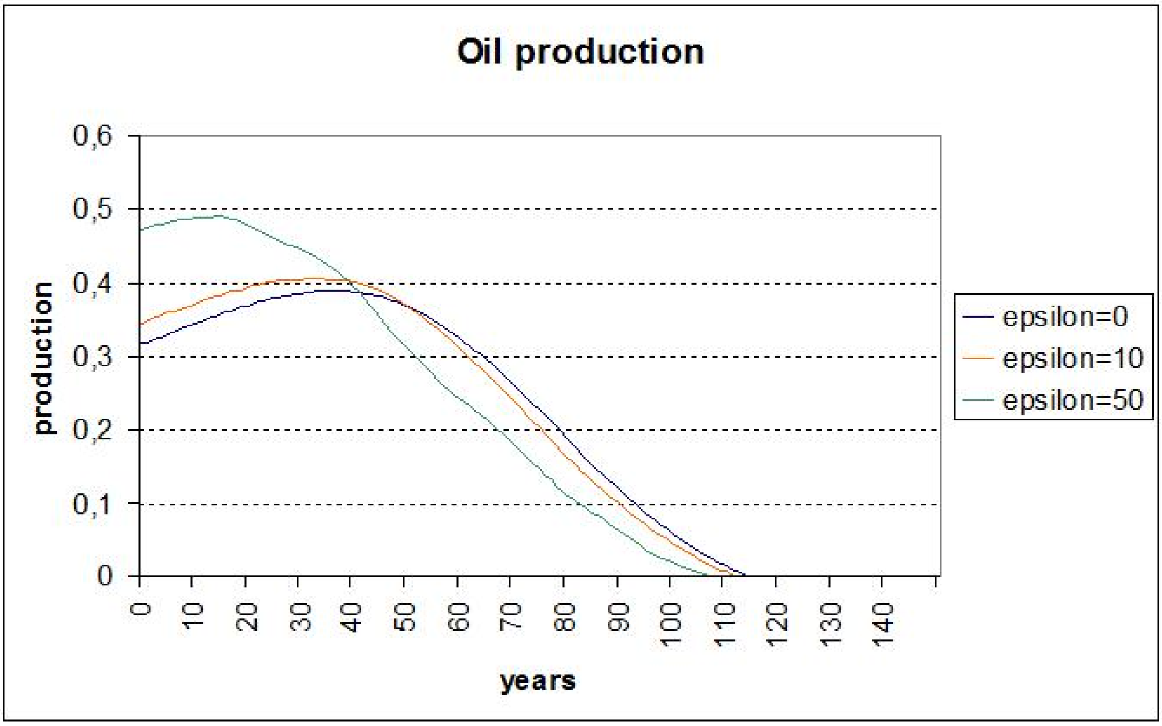

Figure 13: $r = 5\%, \rho = 2\%, \alpha = 10, \beta = 100, \sigma = 1.2$, various ϵ

4.2.3 Comments

We see that ϵ, and hence the ranking effect, has a lot of influence on the shape of the production scheme. As expected, the higher the ϵ, the faster the production.

Although it can change a lot the shape of the production (at least for $\epsilon = 50$) we see that our numerical methodology is robust for a large range of ϵ's. Even though $\epsilon = 50$ represents a very high value, the curve obtained by our algorithm is not too irregular. This is perhaps one of the force of our mean field game method: it allows to solve problems that take account of non-economic interactions between agents and solutions can still be found when the social effects dominates the classical economic ones, changing henceforth drastically the behavior of agents.

5 The two-energy case

In this section, we consider that there is, in addition to oil producers, potential entrants on the market. These entrants produce an imperfect substitute to oil but they use a renewable source of energy. Typically, we have in mind the competition between fossil energy and electricity to fuel cars. The reserves of oil are finite whereas electricity can always be produced (or at least there is no such reserve problem for the electricity producers). However, these two products are not exact substitute because autonomy is not the same in both cases for instance. For this reason, the two types of energy producers (oil producers and alternative energy producers) may produce simultaneously.
One of our goal here is to understand when alternative energy producers will enter the market and to discuss the usefulness of a subsidy to induce earlier entry.

5.1 Demand function

To model demand on the energy market we are going to use a classical economic setting for demand functions: if we denote oil by 1 and the alternative energy by 2 then we can postulate the following demand functions:

$$D_1(t, p_1, p_2) = Ee^{\rho t} p_{en}^{-\sigma} \left(\frac{p_1}{p_{en}}\right)^{-\eta}$$

$$D_2(t, p_1, p_2) = Ee^{\rho t} p_{en}^{-\sigma} \left(\frac{p_2}{p_{en}}\right)^{-\eta}$$

where $\eta > \sigma$ and where p_{en} stands for a reference price for energy:

$$p_{en} = \left(p_1^{1-\eta} + p_2^{1-\eta}\right)^{\frac{1}{1-\eta}}$$

These demand functions are justified by CES utility functions (see the box below) and allow us to factor in three effects:

- A wealth effect, through $Ee^{\rho t}$. As before we have an exogenous growth of the economy at constant rate ρ.
- A substitution effect between energy and other goods which is modeled by the term $p_{en}^{-\sigma}$. σ represents here the elasticity of demand for the

energy aggregate.

- A substitution effect between oil and the alternative energy. This effect is modeled by $\left(\frac{p_1}{p_{en}}\right)^{-\eta}$ and $\left(\frac{p_2}{p_{en}}\right)^{-\eta}$ where we see that demand for one or the other energy depends on the relative price of one energy compared to the reference price for energy. Here η stands for the elasticity of substitution between the two types of energy.

<u>Microeconomic foundations of the demand functions:</u>

To justify these demand functions let's consider the general case where we have a continuum of sectors denoted by i and for each sector a continuum of products denoted by j. The CES utility function associated to this framework is:

$$U((X_i^j)_{i,j}) = \int \left[\left(\int X_i^{j\,\frac{\eta-1}{\eta}} dj \right)^{\frac{\eta}{\eta-1}} \right]^{\frac{\sigma-1}{\sigma}} di$$

Hence if we use the aggregate $X_i = \left(\int X_i^{j\,\frac{\eta-1}{\eta}} dj \right)^{\frac{\eta}{\eta-1}}$ we see that the program of an agent with wealth E is characterized by the first order conditions:

$$\forall i, j, \quad \frac{dX_i}{dX_i^j} X_i{}^{-1/\sigma} = \lambda p_i^j$$

$$\Rightarrow X_i^{j\,-1/\eta} X_i{}^{1/\eta-1/\sigma} = \lambda p_i^j$$

Hence, we have

$$\left(\int X_i^{j\,\frac{\eta-1}{\eta}} dj \right) X_i{}^{1/\eta-1/\sigma} = \lambda \int p_i^j X_i^j dj$$

$$\Rightarrow {X_i}^{1-1/\sigma} = \lambda \int p_i^j X_i^j dj$$

Also we have

$$X_i^{j\,\frac{\eta-1}{\eta}} {X_i}^{(1/\eta-1/\sigma)(1-\eta)} = \lambda^{1-\eta} {p_i^j}^{1-\eta}$$

$$\Rightarrow {X_i}^{\frac{\eta-1}{\eta}} {X_i}^{(1/\eta-1/\sigma)(1-\eta)} = \lambda^{1-\eta} \int {p_i^j}^{1-\eta} dj$$

$$\Rightarrow {X_i}^{\frac{\eta-1}{\sigma}} = \lambda^{1-\eta} \underbrace{\int {p_i^j}^{1-\eta} dj}_{{P_i}^{1-\eta}}$$

where P_i is the aggregated price in sector i. Hence:

$${X_i}^{-\frac{1}{\sigma}} = \lambda P_i$$

The real significance of P_i is then given by the combination of our two relations above: $\int p_i^j X_i^j dj = P_i X_i$. Now, ${X_i}^{-\frac{1}{\sigma}} = \lambda P_i \Rightarrow P_i X_i = \lambda^{-\sigma} {P_i}^{1-\sigma}$.

Hence $E = \lambda^{-\sigma} \underbrace{\int {P_i}^{1-\sigma} di}_{P^{1-\sigma}}$ where P is a global price level.

Combining all our results and going back to X_i^j, we see that:

$$X_i^j = \frac{E}{P} \left(\frac{P_i}{P}\right)^{-\sigma} \left(\frac{p_i^j}{P_i}\right)^{-\eta}$$

and this is exactly what we considered.

Now that we have the demand part of the problem, let's turn to the supply side.

5.2 Program of the entrants

We suppose that there is a club of potential entrants that collude to decide when they enter the market. However, we suppose that once they entered the market, they cannot collude on price and that this continuum (of fixed size, here 1) is characterized by a perfect competition hypothesis. This idea of collusion on entry and not on prices can be justified by large fixed costs.

A large number of entrants decide to merge as a club to pay for the network infrastructure that is necessary to enter the market. However, once they have paid the fixed cost to enter, they have no reason to collude and price competition leads to a perfect competition case.
Mathematically, each potential entrant (they are identical to simplify) faces a program of the following form:

$$\max_{T,q_2(\cdot)} -Fe^{\pi T}e^{-rT} + \int_T^{\infty} \left(p_2(t)q_2(t) - C_2(q_2(t))\right) e^{-rt}dt$$

where the notations are similar to those used for oil producers, though we suppose that the parameters of the cost function $C_2(q) = \alpha_2 q + \frac{\beta_2}{2}q^2$ are not the same as for oil to model a cheaper or a more expensive technology and also to model, in some sense[4], the idea of different capacity constraints between the two sectors.
F here is a sunk cost to enter the market and this cost evolves at rate $\pi < r$ to model inflation. T is the time to enter the market. Because of collusion, T chosen individually will be the same as T chosen by the club and that's the reason why each agent optimizes on T.

We can characterize the time at which entrants arrive on the market:

Proposition 5.1 (Time of entry)**.** *Entrants turn up at time T when the post-entry price after time T of their alternative energy verifies:*

$$p_2(T) = \alpha_2 + \sqrt{2\beta_2(r-\pi)Fe^{\pi T}}$$

Proof:

First of all, once the entrants are on the market, the price $p_2(t)$ is simply given by the marginal cost of production. Hence, production is given by $q_2(t) = \frac{p_2(t)-\alpha_2}{\beta_2}$. Consequently, the first order condition for T is:

$$F(r-\pi)e^{-(r-\pi)T} = e^{-rT}\frac{1}{2\beta_2}(p_2(T)-\alpha_2)^2$$

[4] This is one of the roles played by the quadratic term

$$\Rightarrow p_2(T) = \alpha_2 + \sqrt{2\beta_2(r-\pi)Fe^{\pi T}}$$

□

There can *a priori* be several such T's. In the simulations we are going to present later on, uniqueness will be true empirically.

5.3 Analysis of the model

We can consider any of the preceding oil production models and plug it into the 2-energy framework. All the resulting models are of the same kind when it comes to analyzing the impact of potential entrants.
We have seen before that entrants decide to enter the market when the post-entry equilibrium price[5] of their alternative energy is sufficiently high. Oil producers know this rule and adapt their strategies to this new context. Since entrants steal a part of the energy demand to oil producers, oil producers may want the entrants to arrive on the market as late as possible. To do that, they have to produce less than it would be the case if they were alone: using this strategy they keep enough oil in the ground so that, after an hypothetical entry, prices are low. This effect is especially relevant when a monopoly produces oil instead of a competitive continuum because oil producers keep reserves high to threaten the potential entrant of massive production after entry; a massive production that induces low prices and hence deter entry. On the other hand, and this is particularly relevant when agents are atomized and hence cannot play complex strategies that directly influence prices, each producer may want to get rid of his oil reserve, not to suffer from the competition of the entrant. In practice, there is a trade-off between these two effects and we can only analyze what happens on numerical examples.

5.4 Numerical results and effect of a subsidy

Let's consider that there is no inflation and that the two sources of energy are characterized by $(\alpha_1, \beta_1) = (10, 100)$ for oil and $(\alpha_2, \beta_2) = (50, 50)$ for the

[5] An uncorrect reasoning would be to consider the prices of oil just before entry.

alternative energy. If the fixed cost to enter is equal to 50 then the evolution of oil production is the one represented on Figure 14.

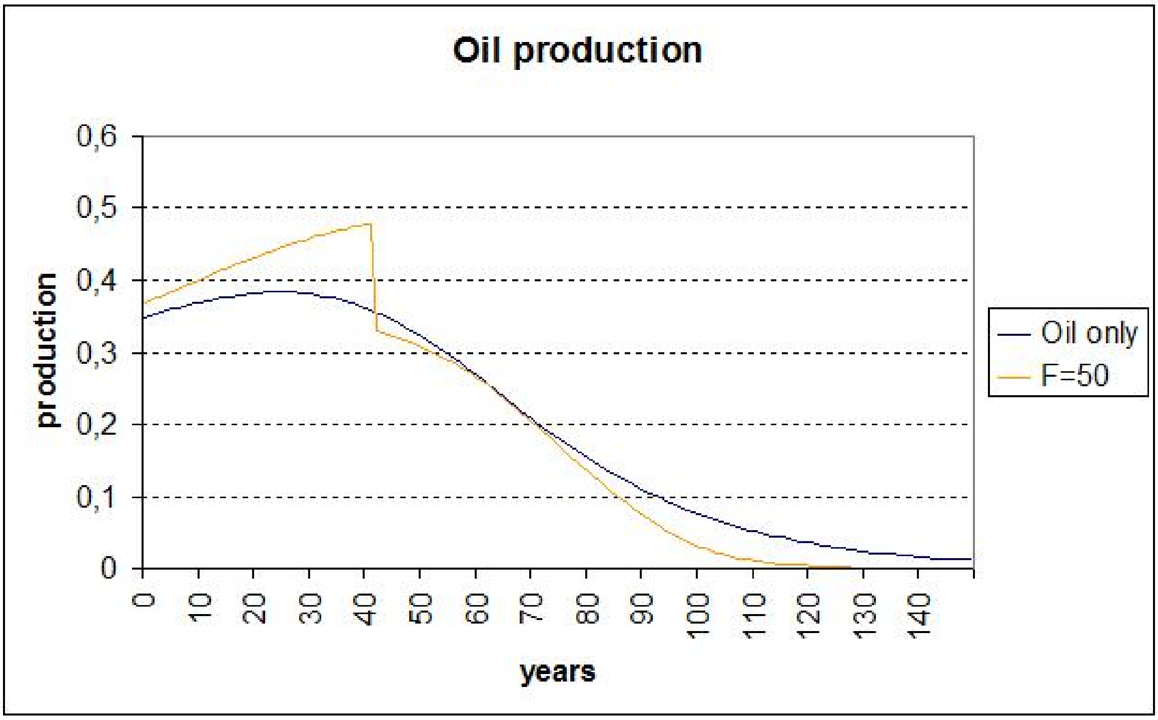

Figure 14: Evolution of the total oil production

We see that the second effect dominates in the above trade-off. Because of their rational expectations, oil producers know when their competitors are going to enter and they individually find it optimal to get rid of their oil reserves more rapidly than before. Hence, the very presence of potential entrants implies an increase in production and hence an increase in pollution for the first 40 years in our example.
This could raise the idea of a subsidy to the alternative energy producers in order to diminish this side effect that may seem absurd since, in some sense non-polluting alternative energies... pollute, though not directly.

Let's for instance consider the case of $F = 50$ and then consider a subsidy[6] of 25 to reduce F to 25. We obtain the evolution for oil production of Figure 15.

Here, the difference of pollution between the two cases $F = 50$ and $F = 25$ is very low during the first 30 years, though the subsidy induces slightly higher pollution, but the pollution last a shorter period of time and a subsidy is arguably welfare-enhancing for any reasonable welfare function that model the damage or disutility of pollution (see below the disutility figures).

[6] In our analysis, we do not focus on the origin of the subsidy. One should have in mind that our model does not take into account a subsidy that would be financed with complex tax mechanisms on oil prices. To simplify, we decided to consider an exogenous subsidy from the state, as it is often the case to subsidize sustainable projects, or a subsidy financed through a non-distortive tax such as a tax on oil companies profits.

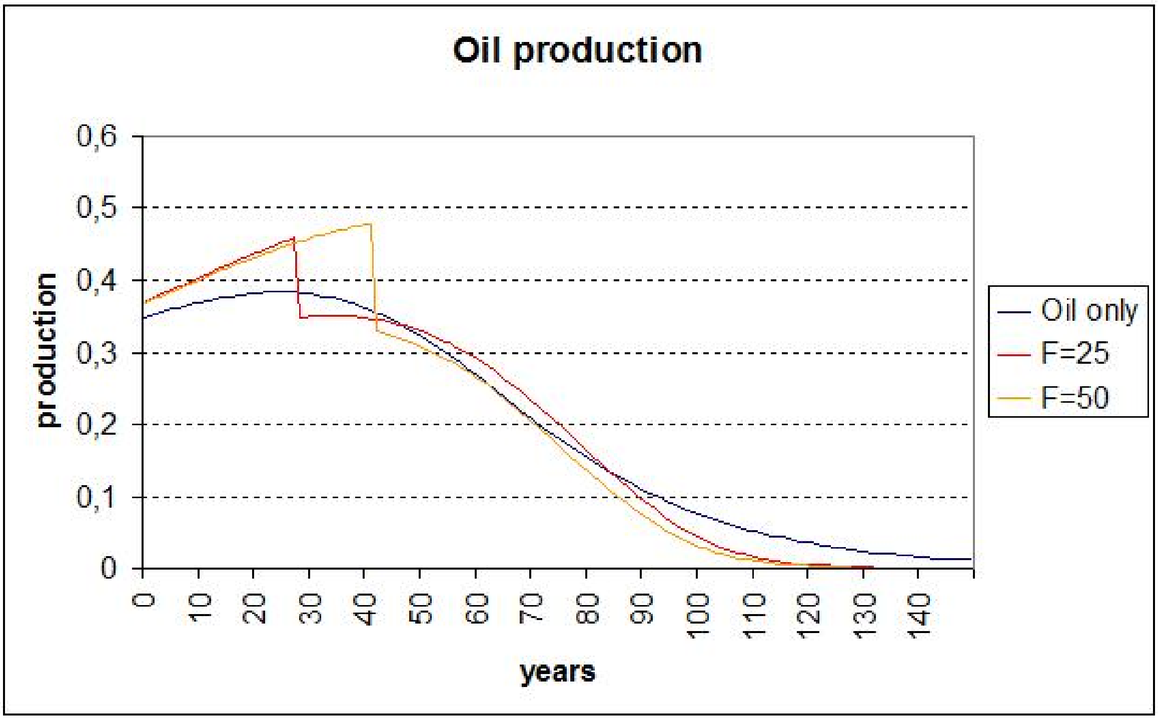

Figure 15: Impact of a subsidy from $F = 50$ to $F = 25$

Even better, when the alternative energy is cheap to produce (that is $\alpha_2 = 10$ instead of 50), a subsidy may remove the period of high production as on the example of Figure 16 where F goes from 100 to 50 thanks to the subsidy.

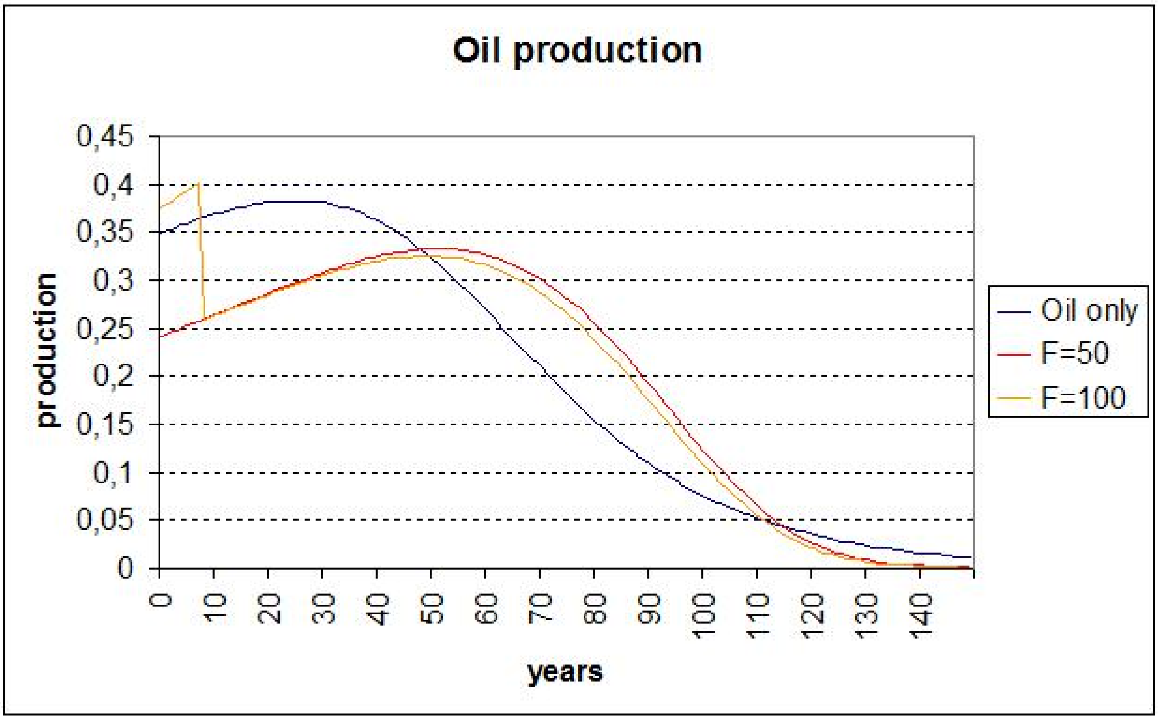

Figure 16: Impact of a subsidy from $F = 100$ to $F = 50$ when the alternative energy is cheap to produce ($\alpha_2 = 10, \beta_2 = 50$)

However, this welfare-enhancing effect of a subsidy may not exist for example if the fixed cost is really high. Imagine for example that the subsidy may not be higher than 100 because of a budget constraint. Then, if $F = 250$, in the case where $\alpha_2 = 50$, a subsidy may be very harmful in terms of pollution for the next 60 years or so because the induced increase in pollution is really high as on Figure 17.

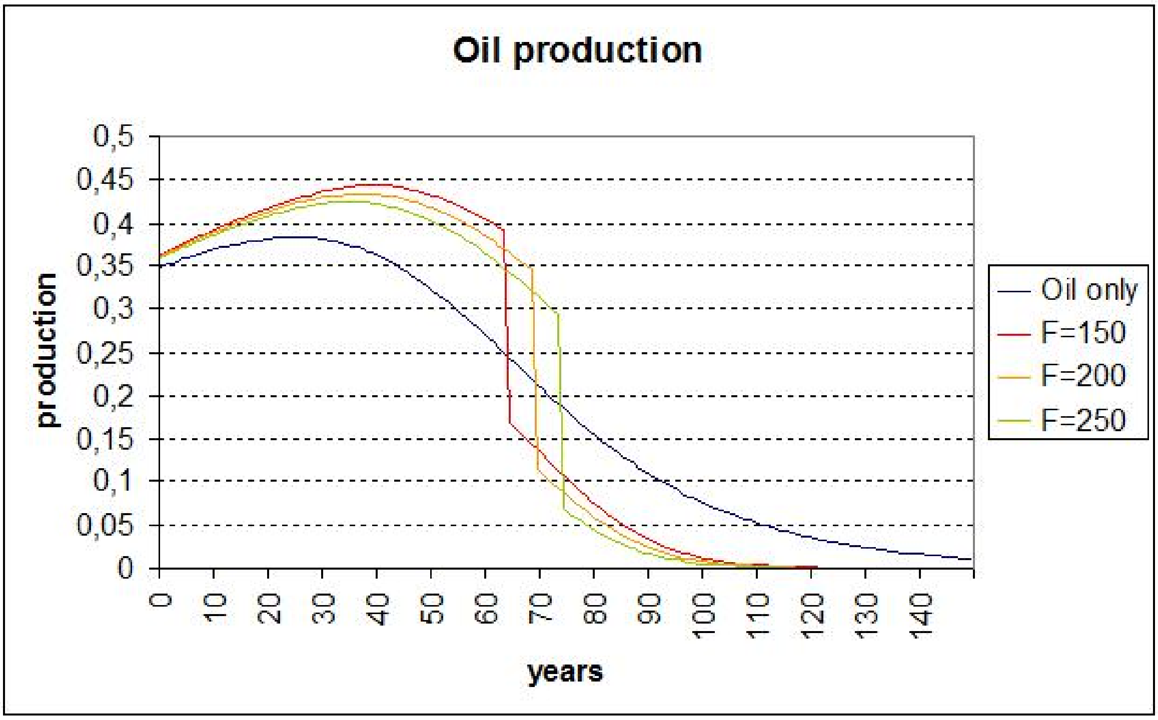

Figure 17: Impact of a subsidy from $F = 250$ to $F = 200$ or $F = 150$

5.5 Conclusion on the two-energy case

5.5.1 Disutility of pollution

This two-energy case allowed us to discuss briefly the impact of a subsidy to alternative energy producers. So far, we just reasoned on graphs and the reader may not be convinced by our arguments on the benefits or harmfulness of the subsidy. At the end of the day, one may argue that, since all the oil reserves will be exploited the overall cumulated pollution will be the same in all circumstances. However, if we introduce a disutility function for pollution with a reasonable discount rate, it's often better to postpone production of oil and/or to smooth the gas emissions.

Let's consider, as an illustration, a natural family of disutility functions for pollution. Since pollution in the atmosphere at time t is the result of past production, we can write pollution at time t as:

$$\Pi_t = \int_{-\infty}^{t} f(q_s) \exp(-\lambda(t-s)) ds$$

where f stands for the pollution production function and where λ models the fact that pollution does have a half-life (λ is arguably very small). Hence, a disutility function for pollution may be $U = \int_0^\infty \Pi_t \exp(-\xi t) dt$ or equivalently[7], in the sense of disutility functions, $\int_0^\infty f(q_t) \exp(-\xi t) dt$. In these equations, ξ is the discount rate to be applied to environmental goods (see [8, 6] for more details on ecological discount rates).

Hence, if we consider a linear function of production for pollution and various discount rates ξ, we can compare the disutility figures associated to the different graphs presented above.

First we can write the reference case where there are only oil producers:

[7] We permute the integral signs and consider the production to come ($t \geq 0$).

Discount rate ξ	Disutility (oil only)
0	29.49
0.5%	23.95
1%	19.86
2%	14.39
5%	7.29

Now, if we consider the case presented in Figure 15, the following figures[8] show that the subsidy indeed improves welfare in the sense that it reduces disutility for several admissible values of the discount rate (the values only have an ordinal sense and the difference between two figures has no meaning apart from its sign.).

	Disutility	
Discount rate ξ	$\alpha_2 = 50, F = 50$	$\alpha_2 = 50, F = 25$
0	29.47	29.47
0.5%	24.60	24.39
1%	20.82	20.50
2%	15.49	15.12
5%	8.02	7.83

On the contrary, and as announced in the comments regarding Figure 17[9] a subsidy of 50 or 100 is not welfare-enhancing when the alternative energy is relatively expensive and characterized by high fixed cost of entry:

[8]The figures for $\xi = 0$ are supposed to be always the same. The differences that occur are relatively small and linked to numerical errors in the calculations of the optimal production.

[9]The results for Figure 16 are the following:

	Disutility	
Discount rate ξ	$\alpha_2 = 10, F = 100$	$\alpha_2 = 10, F = 50$
0	29.46	29.46
0.5%	23.46	23.12
1%	19.09	18.53
2%	13.38	12.61
5%	6.54	5.65

	Disutility		
Discount rate ξ	$\alpha_2 = 50, F = 250$	$\alpha_2 = 50, F = 200$	$\alpha_2 = 50, F = 150$
0	29.48	29.47	29.48
0.5%	24.66	24.71	24.74
1%	20.87	20.95	21.00
2%	15.45	15.55	15.63
5%	7.83	7.90	7.97

5.5.2 Origin of the subsidy

Our model is arguably simple because we do not explicit where the subsidy is coming from and we do not consider for instance a subsidy financed by taxing oil production as it would be the case in a "carbon tax"-like mechanism. However, as we mentioned it earlier, our model is consistent with either an exogenous subsidy, or a subsidy financed through a tax on oil producers' profits. Our conclusion should be hence understood as follows: a tax on oil producers profits to subsidize alternative energies may be harmful in terms of pollution when the alternative energy is costly to produce and need important initial infrastructure investment (the subsidy being understood as a way to partly finance the said infrastructure - network infrastructure for instance). Even though our conclusion may be discussed in a different setups if producers have a market power or if the subsidy comes from a tax on oil prices, the non-classical pollution effect we exhibit invites to reconsider the welfare effects of a subsidy to alternative energy producers, especially when the alternative energy is costly to produce compared to oil.

Conclusion

Starting with an orthodox profit-maximizing framework in a deterministic context, we progressively built the mean field game tools to be used for several generalizations of the basic model. These tools turn out to be very robust for various extensions such as the introduction of randomness or the adding of non-profit (or social) interactions.

In fact the framework we designed is suited to even more general settings an economist would like to use. For instance, using 2 Bellman functions, one can easily build a Stackelberg model to take account of a big player in the case of oil production (OPEC). One may also want to change the optimization criterion to take account of the research process inside the oil industry to exploit more and more wells. It's also straightforward to change the equations in order to have various types of oil producers or various types of wells (to model heterogeneity in production costs and/or capacity constraints for instance). This extensive study is therefore far from exhaustive and we can easily build upon our models, using, *mutatis mutandis*, our very robust numerical methods.

Appendix

MATLAB source codes for the determination of an equilibrium in the deterministic model:

- calcboucle.m: this part of the code takes a price function $p(t)$ (represented by *tabpr*) and a λ function (represented by *tablamb*) and builds the corresponding production function $q(t,R)$. Once the production function is built, we update the function $t \mapsto p(t)$ and $R \mapsto \lambda(R)$ in order to reduce the gap between supply and demand and between the actual and implied reserves.
  This update is made with a logarithmic eductive scheme.

```
function[new_tabq,new_tabpr,new_tablamb]=
    calcboucle(pas_temps,T,pas_R,long_R,tabq,tabpr,tablamb,
    alpha,beta,rate,rho,pas_theta,m0)

E=40; % For this example we take E=40.

new_tabq=zeros(1+long_R/pas_R,1+T/pas_temps);
new_tabpr=zeros(1,1+T/pas_temps);
new_tablamb=zeros(1,1+long_R/pas_R);

somme1=zeros(1,1+long_R/pas_R);
somme2=zeros(1,1+T/pas_temps);

for R=1:(1+long_R/pas_R)
    for t=1:(1+T/pas_temps)
        new_tabq(R,t)=(max((tabpr(1,t)-alpha-tablamb(1,R)
            *exp(rate*(t-1)*pas_temps)),0))/beta;
        somme1(1,R)=somme1(1,R)+new_tabq(R,t)*pas_temps;
        % We compute in somme1 the total reserve implied
        % by the production scheme.
        % It must be equal to R at equilibrium
```

```
    end;
end;

for t=1:(1+T/pas_temps)
    for R=1:(1+long_R/pas_R)
        somme2(1,t)=somme2(1,t)+new_tabq(R,t)*m0(1,R);
        % We compute the total oil supply at time t
    end;
end;

for t=1:(1+T/pas_temps)
    intermed1 = (1+abs(E*(exp(rho*(t-1)*pas_temps)
        *(tabpr(1,t)^(-1.2))))+abs(somme2(1,t)));
    intermed2 = pas_theta*(E*(exp(rho*(t-1)*pas_temps)
        *(tabpr(1,t)^(-1.2)))-somme2(1,t))/intermed1;
    new_tabpr(1,t)=tabpr(1,t)+tabpr(1,t)
        * min(max(-0.02,intermed2),0.02);
    % p(t) is updated to reduce the gap between supply and demand
    % through a logarithmic eductive scheme
    % with a maximum move of 2 percent
end;

for R=1:(1+long_R/pas_R)
    intermed1= 0.1*pas_theta*(somme1(1,R)-(R-1)*pas_R)
        /(1+abs(somme1(1,R))+abs((R-1)*pas_R));
    new_tablamb(1,R)=tablamb(1,R)+ tablamb(1,R)
        *min(max(-0.02,intermed1),0.02);
    % lambda(R) is updated to reduce the gap between
    % actual and implicit reserves through a logarithmic
    % eductive scheme with a maximum move of 2 percent
    % The speed of the eductive scheme is one tenth
    % of the speed of the preceding
end;
```

- resolution_deterministe.m: this function is the main part of the code. It takes as inputs the parameters of the model, the discretization parameters (for time, reserve and for the eductive scheme) and an initial guess for the price function. It returns the result of the eductive scheme after *theta* reached the value *long_theta*.

```
function [tabq,tabpr,tablamb,tabprst,tablambst,tabqst,m0]=
    resolution_deterministe(pas_temps,T,pas_R,long_R,pas_theta,
    long_theta,alpha,beta,rate,rho,p)

prix=zeros(1,1+T/pas_temps);
lambda=ones(1,1+long_R/pas_R);

% First guess

for i=1:(1+T/pas_temps)
    prix(1,i)=p;
end;

% initial distribution of the reserves

m0=zeros(1,1+long_R/pas_R);
somme_m0=0;
for i=floor((1 + long_R/pas_R)/5):(1+long_R/pas_R)
    m0(1,i)=(i-floor((1 + long_R/pas_R)/5))
        *(1+long_R/pas_R-i);
    % We take a parabolic distribution between
    % 20 and 100 percent of the reserves values considered
    somme_m0=somme_m0+m0(1,i);
end;
m0 = m0 / somme_m0;

% Initialization of the variables
```

```
tabq=zeros(1+long_R/pas_R,1+T/pas_temps);
tabpr=zeros(1,1+T/pas_temps);
tablamb=zeros(1,1+long_R/pas_R);
tablamb(1,:)=lambda;
tabpr(1,:)=prix;

% Declaration of the variables to stock the evolution
% of the eductive scheme
% A value is saved every 1000 steps

z = ceil((long_theta/pas_theta) / 1000);
tabqst=zeros(1+z,1+long_R/pas_R,1+T/pas_temps);
tabprst=zeros(1+z,1+T/pas_temps);
tablambst=zeros(1+z,1+long_R/pas_R);

% Eductive scheme

for theta=1:(long_theta/pas_theta)
   [tabq,tabpr,tablamb]=
        calcboucle(pas_temps,T,pas_R,long_R,tabq,tabpr,
        tablamb,alpha,beta,rate,rho,pas_theta,m0);
   % An iteration is made

   % Save the current result
   if theta/1000 == floor(theta/1000)
       for R=1:(1+long_R/pas_R)
           for t=1:(1+T/pas_temps)
               tabqst(theta/1000,R,t)=tabq(R,t);
           end;
           tablambst(theta/1000,R)=tablamb(1,R);
       end;
       for t=1:(1+T/pas_temps)
           tabprst(theta/1000,t)=tabpr(1,t);
       end;
```

```
        % Print theta
    theta
    end;
end;
```

- lanceur_deterministe.m: this script illustrates how to use the preceding functions to obtain the prices and the total production.

```
warning off;
[tabq,tabpr,tablamb,tabprst,tablambst,tabqst,m0]=
    resolution_deterministe(1,150,1,50,0.01,5000,
    10,100,0.05,0.02,50);
% The problem is considered over 150 years with an
% interest rate equal to 5 percent
% and a psychological discount rate equal to 2 percent.
tabpr % Print the values of the function p(t)
tabpr(1,1:50) % Print the prices for the next 50 years
m0 % Print the initial distribution of the reserves
m0*tabq % Print the total production
```

MATLAB source codes for the determination of an equilibrium in the stochastic model:

- resolution_stochastique_complexe.m: this part of the code considers the optimization problem of an agent for a given path of the price p. Due to our method of continuation, the function takes as input not only the parameters of the model and the price p at stake but also the continuation parameter that corresponds to the parameter δ in the above text. The output are the Bellman function, the optimal control q, the evolution of the distribution m that solves the Kolmogorov equation and hence the total supply. This total supply "defines" a price function on which the next function is going to consider an eductive scheme.

```
function [u,q,m,TQ,p]=resolution_stochastique_complexe(
    p_guess,m0,pas_temps,T,pas_R,long_R,sigma,alpha,
```

```
    beta,rate,rho,E,continuation,gaussienne)

u = zeros(1+long_R/pas_R,1+T/pas_temps);
q = zeros(1+long_R/pas_R,1+T/pas_temps);
TQ = zeros(1,1+T/pas_temps);
p = zeros(1,T/pas_temps);
m = zeros(1+long_R/pas_R,1+T/pas_temps);

for t= T/pas_temps:-1:1

    % We compute the convolution product of u
    % and the gaussian to replace the Laplace operator
    uconv = zeros(1+long_R/pas_R,1);
    for R=2:long_R/pas_R+1
        N = floor((3 * sigma* (R-1) * sqrt(pas_temps)));
        uconv(R,1)=u(R,t+1)*gaussienne(R,N+1);
        for j=1:N
            if R+j <= long_R/pas_R+1
                uconv(R,1) = uconv(R,1)
                    + u(max(R-j,1),t+1)* gaussienne(R,N+1+j)
                    + u(R+j,t+1)* gaussienne(R,N+1-j);
            else % Neumann condition in R_max
                uconv(R,1) = uconv(R,1)
                    + u(max(R-j,1),t+1)* gaussienne(R,N+1+j)
                    + u(2*(long_R/pas_R+1)-(R+j),t+1)
                    * gaussienne(R,N+1-j);
            end;
        end;
    end;



    % Optimization at date t
    for R=2:long_R/pas_R+1
```

```
argmax_Rj = R;
max_u = (1-rate * pas_temps)*uconv(R,1);
Rj=R-1;
nv_u = ((p_guess(t)-alpha) * (R-Rj)*pas_R
        - beta/(2*pas_temps) * ((R-Rj)*pas_R)^2)
        + (1- rate * pas_temps)*uconv(Rj,1);
% Remember the cost is a function of the instantaneous
% production

while (nv_u > max_u) & (Rj>= 1)
        max_u = nv_u;
        argmax_Rj = Rj;
        Rj = Rj -1;
        nv_u = ((p_guess(t)-alpha) * (R-Rj)*pas_R
                - beta/(2*pas_temps) * ((R-Rj)*pas_R)^2)
                + (1- rate * pas_temps)*uconv(max(Rj,1),1);
end;

if argmax_Rj == 1
    u(R,t) = max_u;
    q(R,t) = (R-argmax_Rj)*pas_R;
else
    % We find the maximum on the interpolation parabola
    beta_parabole = max_u;
    Rj=argmax_Rj;
    alpha_parabole = ((p_guess(t)-alpha)*(R-(Rj-1))*pas_R
                    - beta/(2*pas_temps)*((R-(Rj-1))*pas_R)^2)
                    + (1-rate*pas_temps)*uconv((Rj-1),1);
    gamma_parabole = ((p_guess(t)-alpha) * (R-(Rj+1))*pas_R
                    - beta/(2*pas_temps)*((R-(Rj+1))*pas_R)^2)
                    + (1-rate*pas_temps)
                    * uconv(min((Rj+1),long_R/pas_R+1),1);

    coeff_a = (gamma_parabole -2*beta_parabole + alpha_parabole)
```

```
                / (pas_R)^2 / 2;
        coeff_b = (gamma_parabole - alpha_parabole) / pas_R / 2
                - 2* coeff_a * (Rj-1)*pas_R;
        coeff_c = beta_parabole
               - coeff_a * ((Rj-1)*pas_R)^2 - coeff_b * (Rj-1)*pas_R;

        if coeff_a == 0
            u(R,t) = max_u;
            q(R,t) = (R-argmax_Rj)*pas_R;
        elseif (-coeff_b/(2*coeff_a) > (Rj-1)*pas_R)
                % The parabola is not suited
            u(R,t) = max_u;
            q(R,t) = (R-argmax_Rj)*pas_R;
        elseif (-coeff_b/(2*coeff_a) < (Rj-2)*pas_R)
                % The parabola is not suited
            u(R,t) = max_u;
            q(R,t) = (R-argmax_Rj)*pas_R;
        else
            q(R,t) = (R - 1)*pas_R - (-coeff_b/(2*coeff_a));
            u(R,t) = -coeff_b^2 / coeff_a / 4 + coeff_c;
        end;
    end;
  end;
end;

% Initialization of m

m(:,1) = m0;

% Transport equation for m

 for t=1:T/pas_temps
     TQ(1,t) = sum(q(:,t).*m(:,t));
     for R=2:1+long_R/pas_R
```

```
    cible = R-q(R,t)/pas_R;
    arr_inf = floor(R-q(R,t)/pas_R);
    arr_sup = arr_inf + 1;
    m(max(arr_inf,1),t+1) = m(max(arr_inf,1),t+1)
       + (arr_sup - cible) * m(R,t);
    m(arr_sup,t+1) = m(arr_sup,t+1)
       + (cible - arr_inf) * m(R,t);
end;

% We now apply the convolution
% instead of the Laplace operator
mconv = zeros(1+long_R/pas_R,1);

for R=2:long_R/pas_R+1
    N = floor((3 * sigma* (R-1) * sqrt(pas_temps)));

    mconv(R,1)=mconv(R,1)+m(R,t+1)*gaussienne(R,N+1);
    for j=1:N
        if (R+j <= long_R/pas_R+1)
            mconv(R-j,1) = mconv(R-j,1)
               + m(R,t+1)* gaussienne(R,N+1-j);
            mconv(R+j,1) = mconv(R+j,1)
               + m(R,t+1)* gaussienne(R,N+1+j);
        else % Neumann condition in R_max
    mconv(R-j,1) = mconv(R-j,1)
               + m(R,t+1)* gaussienne(R,N+1-j);
    mconv(2*(long_R/pas_R+1)-(R+j),1) =
               mconv(2*(long_R/pas_R+1)-(R+j),1)
               + m(R,t+1)* gaussienne(R,N+1+j);
        end;
    end;

end;
m(:,t+1) = mconv(:,1);
```

```
    % Determination of the price with
    % the continuation parameter
    p(1,t) = ((TQ(1,t)+continuation)
           /(E*exp(rho*(t-1)*pas_temps)))^(-1/1.2);
end;
```

- boucle_sto_continuation.m: this function finds for a given level of the continuation parameter, the equilibrium p and hence the solution of the problem, that is (u, q, m). It uses a very slow scheme of convergence toward the fixed point $t \mapsto p(t)$.

```
function [u,q,m,TQ,p,theta]= boucle_sto_continuation(
                            p_guess,m0,pas_temps,T,
                            pas_R,long_R,sigma,alpha,
                            beta,rate,rho,E,continuation,
                            gaussienne)

nb_theta = 90;

u = zeros(1+long_R/pas_R,1+T/pas_temps);
q = zeros(1+long_R/pas_R,1+T/pas_temps);
TQ = zeros(nb_theta,1+T/pas_temps);
p = zeros(nb_theta,T/pas_temps);
p_guess_un = zeros(1,T/pas_temps);
p_guess_un = p_guess;

pg = zeros(1,T/pas_temps);
m = zeros(1+long_R/pas_R,1+T/pas_temps);

[u,q,m,TQtheta,ptheta]=resolution_stochastique_complexe(
                       p_guess_un,m0,pas_temps,T,pas_R,
                       long_R,sigma,alpha,beta,rate,rho,
                       E,continuation,gaussienne);
```

```
TQ(1,:) = TQtheta;
p(1,:) = ptheta;
pg = p_guess_un;

if continuation == 1
    p_guess_un = ptheta; % This case is here for a future use
else
    for i=1:T/pas_temps
        % update the price slowly
        p_guess_un(i) = p_guess_un(i)
        + max(-0.05*p_guess_un(i),
        min(0.05*p_guess_un(i),(ptheta(i) - p_guess_un(i))/6));
    end;
end;

theta = 2
erreur = 100;
nv_erreur = max(abs((ptheta -pg)./pg))
% Print the error between two steps.

while ((nv_erreur > 0.02) && (theta <= nb_theta))
        && (erreur > nv_erreur)
    [u,q,m,TQtheta,ptheta]=resolution_stochastique_complexe(
                           p_guess_un,m0,pas_temps,T,pas_R,
                           long_R,sigma,alpha,beta,rate,rho,
                           E,continuation,gaussienne);
    TQ(theta,:) = TQtheta;
    p(theta,:) = ptheta;
    pg = p_guess_un;
    for i=1:T/pas_temps
        % update the price slowly
        p_guess_un(i) = p_guess_un(i)
        + max(-0.05*p_guess_un(i),
        min(0.05*p_guess_un(i),(ptheta(i) - p_guess_un(i))/6));
```

```
    end;
    erreur = nv_erreur;

    theta = theta +1
    nv_erreur = max(abs((ptheta -pg)./pg))
end;

if (nv_erreur >= erreur)
theta = max(theta-2,1);
else
theta = theta - 1;
end;
```

- descente_continuation.m: this function starts from a continuation parameter equal to 1 and goes down to the actual value of δ. The initial guess for the price function at each step of the continuation descent is taken as the solution for the preceding continuation parameter.

```
function [u,q,m,TQ,p,table_theta]=descente_continuation(
                                  p_guess,m0,pas_temps,T,
                                  pas_R,long_R,sigma,alpha,
                                  beta,rate,rho,E,gaussienne)

nb_theta = 90;

u = zeros(1+long_R/pas_R,1+T/pas_temps);
q = zeros(1+long_R/pas_R,1+T/pas_temps);
TQ = zeros(nb_theta,1+T/pas_temps);
p = zeros(nb_theta,T/pas_temps);
prix = zeros(1,T/pas_temps);
m = zeros(1+long_R/pas_R,1+T/pas_temps);
table_theta = zeros(1,200);
```

```
A = zeros(T/pas_temps,3);
A(:,1)=ones(T/pas_temps,1);
A(:,2) = pas_temps:pas_temps:T;
A(:,3) = (A(:,2).*A(:,2))/T;

B = zeros(T/pas_temps,T/pas_temps);

B = A * inv(A'*A) * A';

[u,q,m,TQ,p,theta]=boucle_sto_continuation(
                p_guess,m0,pas_temps,T,pas_R,
                long_R,sigma,alpha,beta,rate,
                rho,E,1,gaussienne);

k=1;
table_theta(k) = theta;
plot(p(theta,:));
hold on;
for continuation=0.95:-0.05:0.1
    continuation
  sum(TQ(theta,:))
    [u,q,m,TQ,p,theta]=boucle_sto_continuation(
        p(theta,:),m0,pas_temps,T,pas_R,long_R,sigma,
        alpha,beta,rate,rho,E,continuation,gaussienne);
    k = k + 1;
    table_theta(k) = theta;
    p(theta,:)
end;
```

- lanceur.m: this script launches the continuation descent with the chosen parameters and returns the total production and some values of both the Bellman function u and the distribution m.

```
warning off

for i=1:6001
    m0_6001(i) = max((i-400)*(2001-i),0);
end;

m0_6001 = m0_6001 / sum(m0_6001);

sigma = 0.10;
long_R = 150;
pas_R = 0.025;
pas_temps = 1;
T = 150;
alpha = 10;
beta = 100;
rate = 0.05;
rho = 0.02;
E = 40;

p50 = 50 * ones(1,150);

N = floor((3 * sigma* long_R * sqrt(pas_temps)/pas_R));

gaussienne = zeros(long_R/pas_R+1,1+2*N);

for R=2:long_R/pas_R+1
    % Construct the variable gaussienne for convolutions
    N = floor((3 * sigma* (R-1) * sqrt(pas_temps)));
    if sigma == 0
        gaussienne(R,N+1) = 1;
    else
        gaussienne(R,N+1) = normpdf(0,0,sigma*(R-1)
                            *pas_R* sqrt(pas_temps));
    end;
```

```
    for j=1:N
        gaussienne(R,N+1-j) = normpdf(j*pas_R,0,
                              sigma *(R-1)*pas_R* sqrt(pas_temps));
        gaussienne(R,N+1+j) = normpdf(j*pas_R,0,
                              sigma *(R-1)*pas_R* sqrt(pas_temps));
    end;
    gaussienne(R,:) = gaussienne(R,:) / sum(gaussienne(R,:));
end;


[u,q,m,TQ,p,table_theta]=descente_continuation(
    p50,m0_6001,pas_temps,T,pas_R,long_R,
    sigma,alpha,beta,rate,rho,E,gaussienne);

p'

u(:,1)
u(:,20)
u(:,50)
u(:,100)

m(:,1)
m(:,20)
m(:,50)
m(:,100)

TQ'
```

MATLAB source codes for the algorithm to find equilibria when there is a ranking component in the optimization criterion of the producers (these source codes use the preceding functions to start the continuation descent):

- resolution_stochastique_complexe_classement.m: this part of the code considers the optimization problem of an agent for a given path of the price p and a given m. It is exactly the same code as for the algorithm without a ranking effect except that we take into account the additional term in the Hamilton-Jacobi-Bellman equation using the last evaluation for m.

```
function [u,q,m,TQ,p]=resolution_stochastique_complexe_classement(
                      p_guess,m_guess,m0,pas_temps,T,pas_R,long_R,
                      sigma,alpha,beta,rate,rho,W,continuation,
                      gaussienne,epsilon)

u = zeros(1+long_R/pas_R,1+T/pas_temps);
q = zeros(1+long_R/pas_R,1+T/pas_temps);
TQ = zeros(1,1+T/pas_temps);
p = zeros(1,T/pas_temps);
m = zeros(1+long_R/pas_R,1+T/pas_temps);

% We compute the convolution product of u and
% the gaussian to replace the Laplace operator
for t= T/pas_temps:-1:1
    uconv = zeros(1+long_R/pas_R,1);
    for R=2:long_R/pas_R+1
        N = floor((3 * sigma* (R-1) * sqrt(pas_temps)));
        uconv(R,1)=u(R,t+1)*gaussienne(R,N+1);
        for j=1:N
            if R+j <= long_R/pas_R+1
                uconv(R,1) = uconv(R,1)
                    + u(max(R-j,1),t+1)* gaussienne(R,N+1+j)
                    + u(R+j,t+1)* gaussienne(R,N+1-j);
            else % Neumann condition
                uconv(R,1) = uconv(R,1)
                    + u(max(R-j,1),t+1)* gaussienne(R,N+1+j)
                    + u(2*(long_R/pas_R+1)-(R+j),t+1)
                    * gaussienne(R,N+1-j);
```

```
        end;
    end;
end;


classement = 0;

for R=2:long_R/pas_R+1
    argmax_Rj = R;
    max_u = (1-rate * pas_temps)*uconv(R,1);

    Rj=R-1;

    nv_u = ((p_guess(t)-alpha) * (R-Rj)*pas_R
           - beta/(2*pas_temps)* ((R-Rj)*pas_R)^2)
           + (1- rate * pas_temps)*uconv(Rj,1);

    while (nv_u > max_u) & (Rj>= 1)
            max_u = nv_u;
            argmax_Rj = Rj;
            Rj = Rj -1;
            nv_u = ((p_guess(t)-alpha) * (R-Rj)*pas_R
                   - beta/(2*pas_temps) * ((R-Rj)*pas_R)^2)
                   + (1- rate * pas_temps)*uconv(max(Rj,1),1);
    end;
    classement = classement + m_guess(R,t);

    if argmax_Rj == 1
        u(R,t) = max_u - pas_temps *epsilon *classement;
        q(R,t) = (R-argmax_Rj)*pas_R;
    else % approximation with a parabola
        beta_parabole = max_u;
        Rj=argmax_Rj;
        alpha_parabole = ((p_guess(t)-alpha) * (R-(Rj-1))*pas_R
```

```
                                 - beta/(2*pas_temps) * ((R-(Rj-1))*pas_R)^2)
                                 + (1- rate * pas_temps)*uconv((Rj-1),1);
            gamma_parabole = ((p_guess(t)-alpha) * (R-(Rj+1))*pas_R
                                 - beta/(2*pas_temps) * ((R-(Rj+1))*pas_R)^2)
                                 + (1- rate * pas_temps)
                                 *uconv(min((Rj+1),long_R/pas_R+1),1);

            coeff_a = (gamma_parabole -2*beta_parabole + alpha_parabole)
                      / (pas_R)^2 / 2;
            coeff_b = (gamma_parabole - alpha_parabole) / pas_R / 2
                      - 2* coeff_a * (Rj-1)*pas_R;
            coeff_c = beta_parabole - coeff_a * ((Rj-1)*pas_R)^2
                      - coeff_b * (Rj-1)*pas_R;

            if coeff_a == 0
                u(R,t) = max_u - pas_temps * epsilon * classement;
                q(R,t) = (R-argmax_Rj)*pas_R;
            elseif (-coeff_b/(2*coeff_a) > (Rj-1)*pas_R)
                u(R,t) = max_u - pas_temps * epsilon * classement;
                q(R,t) = (R-argmax_Rj)*pas_R;
            elseif (-coeff_b/(2*coeff_a) < (Rj-2)*pas_R)
                u(R,t) = max_u - pas_temps * epsilon * classement;
                q(R,t) = (R-argmax_Rj)*pas_R;
            else
                q(R,t) = (R - 1)*pas_R - (-coeff_b/(2*coeff_a));
                u(R,t) = - coeff_b^2 / coeff_a / 4 + coeff_c
                         - pas_temps * epsilon * classement;
            end;
        end;
    end;
end;

m(:,1) = m0;
```

```
for t=1:T/pas_temps
    TQ(1,t) = sum(q(:,t).*m(:,t));
    for R=2:1+long_R/pas_R
        cible = R-q(R,t)/pas_R;
        arr_inf = floor(R-q(R,t)/pas_R);
        arr_sup = arr_inf + 1;
        m(max(arr_inf,1),t+1) = m(max(arr_inf,1),t+1)
                          + (arr_sup - cible) * m(R,t);
        m(arr_sup,t+1) = m(arr_sup,t+1)
                          + (cible - arr_inf) * m(R,t);
    end;
    % Convolution
    mconv = zeros(1+long_R/pas_R,1);

    for R=2:long_R/pas_R+1
        N = floor((3 * sigma* (R-1) * sqrt(pas_temps)));

        mconv(R,1)=mconv(R,1)+m(R,t+1)*gaussienne(R,N+1);
        for j=1:N
            if (R+j <= long_R/pas_R+1)
                mconv(R-j,1) = mconv(R-j,1)
                          + m(R,t+1)* gaussienne(R,N+1-j);
    mconv(R+j,1) = mconv(R+j,1)
                          + m(R,t+1)* gaussienne(R,N+1+j);
            else % Neumann
        mconv(R-j,1) = mconv(R-j,1)
                          + m(R,t+1)* gaussienne(R,N+1-j);
        mconv(2*(long_R/pas_R+1)-(R+j),1) =
                    mconv(2*(long_R/pas_R+1)-(R+j),1)
                    + m(R,t+1)* gaussienne(R,N+1+j);
            end;
        end;
    end;
    m(:,t+1) = mconv(:,1);
```

```
    p(1,t) = ((TQ(1,t)+continuation)
       /(W*exp(rho*(t-1)*pas_temps)))^(-1/1.2);
 end;
```

- boucle_sto_continuation_classement.m: this function finds for a given level of the continuation parameter, the equilibrium p and hence the solution of the problem, that is (u, q, m). It uses a very slow scheme of convergence toward the fixed point $t \mapsto p(t)$.

```
function [u,q,m,TQ,p,theta]=boucle_sto_continuation_classement(
                            p_guess,m_guess,m0,pas_temps,T,pas_R,
                            long_R,sigma,alpha,beta,rate,rho,W,
                            continuation,gaussienne,epsilon)

nb_theta = 90;

u = zeros(1+long_R/pas_R,1+T/pas_temps);
q = zeros(1+long_R/pas_R,1+T/pas_temps);
TQ = zeros(nb_theta,1+T/pas_temps);
p = zeros(nb_theta,T/pas_temps);
p_guess_un = zeros(1,T/pas_temps);
p_guess_un = p_guess;

pg = zeros(1,T/pas_temps);
m = zeros(1+long_R/pas_R,1+T/pas_temps);

[u,q,m,TQtheta,ptheta]=
    resolution_stochastique_complexe_classement(
        p_guess_un,m_guess,m0,pas_temps,T,pas_R,long_R,sigma,
        alpha,beta,rate,rho,W,continuation,gaussienne,epsilon);

TQ(1,:) = TQtheta;
p(1,:) = ptheta;
```

```
pg = p_guess_un;
for i=1:T/pas_temps
    p_guess_un(i) = p_guess_un(i)
    + max(-0.05*p_guess_un(i),
    min(0.05*p_guess_un(i),(ptheta(i) - p_guess_un(i))/6));
end;

theta = 2

erreur = 100;
nv_erreur = max(abs((ptheta -pg)./pg))

while ((nv_erreur > 0.02) && (theta <= nb_theta))
                          && (erreur > nv_erreur)
    [u,q,m,TQtheta,ptheta]=resolution_stochastique_complexe_classement(
                           p_guess_un,m_guess, m0,pas_temps,T,pas_R,
                           long_R,sigma,alpha,beta,rate,rho,W,
                           continuation,gaussienne,epsilon);
    TQ(theta,:) = TQtheta;
    p(theta,:) = ptheta;
    pg = p_guess_un;
    for i=1:T/pas_temps
        p_guess_un(i) = p_guess_un(i)
        + max(-0.05*p_guess_un(i),
        min(0.05*p_guess_un(i),(ptheta(i) - p_guess_un(i))/6));
    end;
    erreur = nv_erreur;
    theta = theta +1
    nv_erreur = max(abs((ptheta -pg)./pg))
end;

if (nv_erreur >= erreur)
theta = max(theta-2,1);
else
```

```
theta = theta - 1;
end;
```

- descente_continuation_classement.m: this function starts from a continuation parameter equal to 1 and goes down to the actual value of δ. The initial guess for the price function at each step of the continuation descent is taken as the solution for the preceding continuation parameter. For the first step (continuation=1), calculations are done using the algorithm with no ranking effect, in order to have a relevant first guess for m.

```
function [u,q,m,TQ,p,table_theta]=
        descente_continuation_classement(
            p_guess,m0,pas_temps,T,pas_R,long_R,sigma,
            alpha,beta,rate,rho,W,gaussienne,epsilon)

nb_theta = 90;

u = zeros(1+long_R/pas_R,1+T/pas_temps);
q = zeros(1+long_R/pas_R,1+T/pas_temps);
TQ = zeros(nb_theta,1+T/pas_temps);
p = zeros(nb_theta,T/pas_temps);
prix = zeros(1,T/pas_temps);
m = zeros(1+long_R/pas_R,1+T/pas_temps);
table_theta = zeros(1,200);

A = zeros(T/pas_temps,3);
A(:,1)=ones(T/pas_temps,1);
A(:,2) = pas_temps:pas_temps:T;
A(:,3) = (A(:,2).*A(:,2))/T;

B = zeros(T/pas_temps,T/pas_temps);

B = A * inv(A'*A) * A';
```

```
% We consider the method without
% a ranking to have a first guess for m
[u,q,m,TQ,p,theta]=boucle_sto_continuation(
                   p_guess,m0,pas_temps,T,pas_R,long_R,
                   sigma,alpha,beta,rate,rho,W,1,gaussienne);

k=1;
table_theta(k) = theta;
plot(p(theta,:));
hold on;
for continuation=0.95:-0.05:0.1
    continuation
  sum(TQ(theta,:))
    [u,q,m,TQ,p,theta]=boucle_sto_continuation_classement(
                       p(theta,:),m,m0,pas_temps,T,pas_R,
                       long_R,sigma,alpha,beta,rate,rho,
                       W,continuation,gaussienne,epsilon);
    k = k + 1;
    table_theta(k) = theta;
    p(theta,:)
end;
```

- lanceur.m: this script launches the continuation descent with the chosen parameters and returns the total production and some values of both the Bellman function u and the distribution m.

```
warning off

for i=1:6001
    m0_6001(i) = max((i-400)*(2001-i),0);
end;

m0_6001 = m0_6001 / sum(m0_6001);
```

```
sigma = 0.02;
long_R = 150;
pas_R = 0.025;
pas_temps = 1;
T= 150;
alpha = 10;
beta = 100;
rate = 0.05;
rho = 0.02;
W= 40;
epsilon=50;
p50 = 50 * ones(1,150);

N = floor((3 * sigma* long_R * sqrt(pas_temps)/pas_R));

gaussienne = zeros(long_R/pas_R+1,1+2*N);

for R=2:long_R/pas_R+1
    N = floor((3 * sigma* (R-1) * sqrt(pas_temps)));
    if sigma == 0
        gaussienne(R,N+1) = 1;
    else
        gaussienne(R,N+1) =
            normpdf(0,0,sigma*(R-1)*pas_R* sqrt(pas_temps));
    end;
    for j=1:N
        gaussienne(R,N+1-j) =
            normpdf(j*pas_R,0,sigma *(R-1)*pas_R* sqrt(pas_temps));
        gaussienne(R,N+1+j) =
            normpdf(j*pas_R,0,sigma *(R-1)*pas_R* sqrt(pas_temps));
    end;
    gaussienne(R,:) = gaussienne(R,:) / sum(gaussienne(R,:));
end;
```

```
[u,q,m,TQ,p,table_theta]=descente_continuation_classement(
                         p50,m0_6001,pas_temps,T,pas_R,long_R,sigma,
                         alpha,beta,rate,rho,W,gaussienne,epsilon);

p'

u(:,1)
u(:,20)
u(:,50)
u(:,100)

m(:,1)
m(:,20)
m(:,50)
m(:,100)

TQ'
```

MATLAB source codes for the algorithm to find equilibria with two types of energy:

- The first step is to write a function to solve for (p_1, p_2) the supply/demand system of equations when there are two energies. This is done in the following functions using an eductive scheme for a given function λ: resol2en.m and resol2enbcl.m

```
function[prix1,prix2]=resol2enbcl(
        pr1,pr2,pastheta,pasR,longR,m0,E,rho,sigma,
        eta,alpha1,beta1,alpha2,beta2,lambda,rate,t)

Pen=((pr1^(1-eta))+(pr2^(1-eta)))^(1/(1-eta));
PRi=zeros(2,1);
PRi(1,1)=pr1;
```

```
PRi(2,1)=pr2;
PRf=zeros(2,1);
% We compute the integrals
intder=0;
int=0;
for R=10:(longR/pasR+1)
    int=int+m0(1,R)*
        max(pr1-alpha1-lambda(R,1)*exp(rate*t),0)/beta1;
    if (pr1-alpha1-lambda(R,1)*exp(rate*t))>=0
        intder=intder+m0(1,R)/beta1;
    end;
end;

D=zeros(2,2);  % Jacobian matrix
D(1,1)=E*exp(rho*t)*(-eta*(pr1^(-eta-1))*(Pen^(eta-sigma))
       +(pr1^(-2*eta))*(eta-sigma)*(Pen^(2*eta-sigma-1)))
       -intder;
D(1,2)=E*exp(rho*t)*(eta-sigma)*(Pen^(eta-sigma-1))
        *(pr1^(-eta))*(pr2^(-eta));
D(2,1)=D(1,2);
D(2,2)=E*exp(rho*t)*(-eta*(pr2^(-eta-1))*(Pen^(eta-sigma))
       +(pr2^(-2*eta))*(eta-sigma)*(Pen^(2*eta-sigma-1)))
       -(1/beta2);
% Invert the Jacobian matrix
D1=inv(D);

F=zeros(2,1);
F(1,1)=E*exp(rho*t)*(pr1^(-eta))*(Pen^(eta-sigma))-int;
F(2,1)=E*exp(rho*t)*(pr2^(-eta))*(Pen^(eta-sigma))
        -(pr2-alpha2)/beta2;

PRf=PRi-pastheta*D1*F;
prix1=PRf(1,1);
prix2=PRf(2,1);
```

```
function[pr1,pr2]=resol2en(
    p1,p2,pastheta,pasR,longR,E,rho,sigma,
    eta,alpha1,beta1,alpha2,beta2,lambda,rate,t)

m0=zeros(1,1+longR/pasR);
somo=0;
for i=10:(1+longR/pasR)
    m0(1,i)=(i-10)*(1+longR/pasR-i);
    somo=somo+m0(1,i);
end;
for i=10:(1+longR/pasR)
    m0(1,i)=(m0(1,i)/somo);
end;

pr1=p1;
pr2=p2;
p1sauv1=p1;
p1sauv2=2*p1;
p2sauv1=p2;
p2sauv2=2*p2;
theta=1;
while theta<=5000 &
    (abs(p1sauv2-p1sauv1)>0.00001 | abs (p2sauv2-p2sauv1)>0.00001)
    [pr1,pr2]=resol2enbcl(
        pr1,pr2,pastheta,pasR,longR,m0,E,rho,sigma,
        eta,alpha1,beta1,alpha2,beta2,lambda,rate,t);
    if theta/100==floor(theta/100)
        p1sauv2=p1sauv1;
        p1sauv1=pr1;
        p2sauv2=p2sauv1;
        p2sauv1=pr2;
    end;
    theta=theta+1;
```

```
end;
```

- calculT.m: this function finds, for a given function λ, the optimal time of entry of the second energy producers that corresponds to the time at which the price of oil, if there were 2 sectors, would reach the threshold found in the above text. In this function, we suppose that there is no inflation.

```
function[T]=calculT(
    p1,p2,F,pastps,pastheta,pasR,longR,E,rho,sigma,
    eta,alpha1,beta1,alpha2,beta2,lambda,rate)

[pr1,pr2]=resol2en(
          p1,p2,pastheta,pasR,longR,E,rho,sigma,
          eta,alpha1,beta1,alpha2,beta2,lambda,rate,0);

T=0;
seuil=alpha2+sqrt(2*beta2*rate*F);
while (pr2<seuil) & (T<200)
    T=T+pastps;
    [pr1,pr2]=resol2en(
              p1,p2,pastheta,pasR,longR,E,rho,sigma,eta,
              alpha1,beta1,alpha2,beta2,lambda,rate,T);
end;
```

- calculprixinf.m: this function finds, for a given function λ, the price path if there were only oil on the market.

```
function[pr1]=calculprixinf(p1,t,pastheta,pasR,longR,
                E,rho,sigma,alpha1,beta1,lambda,rate)
m0=zeros(1,1+longR/pasR);
somo=0;
for i=10:(1+longR/pasR)
```

```
    m0(1,i)=(i-10)*(longR/pasR+1-i);
    somo=somo+m0(1,i);
end;
for i=10:(longR/pasR+1)
    m0(1,i)=(m0(1,i)/somo);
end;

    pr1=p1;
    p1sauv1=p1;
    p1sauv2=2*p1;
    theta=1;
    while (theta<5000) & (abs(p1sauv2-p1sauv1)>0.00001)
        int=0;
        intder=0;
        for R=10:(1+longR/pasR)
             int=int+m0(1,R)
                 *max(pr1-alpha1-lambda(R,1)*exp(rate*t),0)/beta1;
             if (pr1-alpha1-lambda(R,1)*exp(rate*t))>=0
                intder=intder+m0(1,R)/beta1;
             end;
        end;

        pr1=pr1-pastheta/(-E*exp(rho*t)*sigma*(pr1^(-sigma-1))-intder)
                *(E*exp(rho*t)*(pr1^(-sigma))-int);
        if theta/100==floor(theta/100)
          p1sauv2=p1sauv1;
          p1sauv1=pr1;
        end;
        theta=theta+1;
    end;
```

- calculeq2en.m: this function uses an eductive algorithm to find a fixed point for the function λ. For a given function λ, the corresponding time of entry is computed. Then, prices and hence total oil production are calculated. Using the constraint on oil reserves, λ is updated and is supposed to converge towards a fixed point.

```
function[prix1,prix2,Q1,lambda1]=calculeq2en(
    p1,p2,pastheta,longtheta,pastps,per,pasR,longR,E,
    rho,sigma,eta,alpha1,beta1,alpha2,beta2,lambda,rate,F)

prix1=zeros(1+per/pastps,1+longtheta/pastheta);
prix2=zeros(1+per/pastps,1+longtheta/pastheta);
Q1=zeros(1+per/pastps,1+longR/pasR);
lambda1=zeros(1+longR/pasR,1+longtheta/pastheta);
lambda2=zeros(1+longR/pasR,1);
lambda2=lambda';
for theta=1:(1+longtheta/pastheta)
theta
    T=calculT(p1,p2,F,pastps,0.01,pasR,longR,E,rho,
              sigma,eta,alpha1,beta1,
              alpha2,beta2,lambda2,rate);
    beep
    for t=1:(T/pastps)
        prix1(t,theta)=calculprixinf(
                p1,(t-1)*pastps,0.01,pasR,longR,E,
                rho,sigma,alpha1,beta1,lambda2,rate);
        for R=10:(1+longR/pasR)
            Q1(t,R)=max(prix1(t,theta)-alpha1-lambda2(R,1)
                        *exp(rate*(t-1)*pastps),0)/beta1;
        end;
    end;
    beep
    for t=(1+T/pastps):(1+per/pastps)
        [A,B]=resol2en(
            p1,p2,0.01,pasR,longR,E,rho,sigma,eta,alpha1,
            beta1,alpha2,beta2,lambda2,rate,(t-1)*pastps);
```

```
        prix1(t,theta)=A;
        prix2(t,theta)=B;
        for R=10:(longR/pasR+1)
            Q1(t,R)=max(prix1(t,theta)-alpha1-lambda2(R,1)
                        *exp(rate*(t-1)*pastps),0)/beta1;
        end;
    end;
    beep
   for R=10:(1+longR/pasR)
       intt=0;
       for t=1:(1+per/pastps)
           intt=intt+Q1(t,R)*pastps;
       end;
       lambda2(R,1)=max(lambda2(R,1)*(1-pastheta*((R-1)*pasR-intt)
                                    /(1+(R-1)*pasR+intt)),0);
       lambda1(R,theta)=lambda2(R,1);
   end;
end;
```

- lanceur_2_energies.m: this script launches the deterministic algorithm with oil only. This is done to obtain a first guess for the function λ. This first guess is then used to find the equilibrium in the presence of potential competitors whose characteristics are chosen here as in one of the example pictured in the main part of this document.

```
warning off

[tabq,tabpr,tablamb,tabprst,tablambst,tabqst,m0]=
    resolution_deterministe(1,150,1,50,0.01,500,
                            10,100,0.05,0.02,50);

(m0*tabq)'
```

```
F=100
a2=10

[prix1,prix2,Q1,lambda1]=
    calculeq2en(50,50,0.02,30,1,150,1,50,40,0.02,
                1.2,3.6,10,100,a2,50,tablamb,0.05,F);

prix1
prix2
Q1
lambda1
```

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